INTRODUCTION

Naturally occurring low-density cellular or porous materials such as wood, honeycomb and bone have long provided inspiration to designers of novel engineering micro-structures. Honeycomb sandwich structures made from either metal or plastic have long been used in aerospace applications because of their light weight and high stiffness. Polymeric foams, another type of porous material, are widely being used in non-structural applications because of their intrinsic properties such as light weight, designability and good noise/vibration isolation/attenuation characteristics. Foams in general can undergo large recoverable deformations compared to solid materials, and therefore can function as cushioning or buffering components. Crushable polymeric and metallic foams are highly nonlinear in nature and are typically characterized by a plasticity-like stress plateau associated with large compressive deformation that is ideally suited to buffer impact loads (1). For specific applications in impact safety and crashworthiness analysis, the stress-strain relationship of these materials must be well understood so as to provide required inputs in finite element models. These codes use material models that are derived on the basis of empirically obtained stress-strain relationships at a specific strain rate or impact velocity (2, 3). Such investigations have contributed to simple design rules that provide guidelines for the design of automobile interior components to enhance occupant protection. However, these simulated responses of foam models could only represent only a limited range of behaviors under specific loading conditions.

In developing an efficient impact buffering system, it is desirable to conduct a complete analysis of the design so as to ensure that a component meets the desired regulations. Subsequently, design, fabrication and testing processes require several iterations before an optimized design is chosen. Finite element modeling and analysis of the candidates are typically employed to reduce the time and cost in the prototype process. Complete material properties and the stress-strain relationship that covers both compressive and tensile responses for a range of strain rates and temperatures are needed to characterize the mechanical behavior. Since polymeric foams are highly nonlinear materials that undergo large deformation in crashworthiness-related cases, it is desirable that the mechanical properties be calculated from a single constitutive equation that describes the complete stress-strain behavior under various loading conditions.

In the following, after presenting a brief review of the existing constitutive models, a phenomenological constitutive model, which is continuously differentiable and defined in the entire real domain such that both the compressive and the tensile stress-strain responses can be characterized through a single-variable multi-parameter function, is presented and validated.
REVIEW OF EXISTING CONSTITUTIVE MODELS

Experimental characterization of mechanical properties of polymeric foam materials for potential automotive applications has been conducted by several researchers (4–7). Some nonlinear empirical stress-strain models have been shown to be useful in capturing the mechanical behavior of various structural foams (8–13). One widely accepted work on this topic was presented by Rusch (8) where the relationship between compressive stress and strain was expressed as

$$\sigma = Ef(\varepsilon), \quad (1)$$

where $E$ is the initial compressive modulus of the foam and $f(\varepsilon)$ is a nonlinear strain function. Although this model represents the loading characteristics of rigid foams reasonably well, it does not consider the effects of the strain rate and temperature. Meinecke and Schwaber (9) included strain rate dependence of compressive modulus in the above equation and proposed the following model

$$\sigma = E(\dot{\varepsilon})f(\varepsilon). \quad (2)$$

Nagy et al. (10) modified the above model by replacing the modulus term using a coupled strain and strain rate function as

$$M(\varepsilon, \dot{\varepsilon}) = \left( \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} \right)^{b_1 + b_2 \varepsilon}, \quad (3)$$

where $b_1$ and $b_2$ are empirical constants and $\dot{\varepsilon}_0$ is a reference strain rate.

All the above constitutive models were verified to be applicable to foams with a given density, but cannot be used to characterize the effect of density. Sherwood et al. (11) combined the effects of temperature, density, strain and strain rate as

$$\sigma = H(T)G(\rho)M(\varepsilon, \dot{\varepsilon})f(\varepsilon), \quad (4)$$

where $H(T)$ and $G(\rho)$ are functions of temperature and initial density, respectively, and $f(\varepsilon)$ is a shape function. All these functions can be approximated from experimentally obtained data. Chou et al. (12) coupled temperature and strain rate effects in the above model as

$$\sigma = H(\dot{\varepsilon}, T)G(\rho)f(\varepsilon), \quad (5)$$

and verified the model on polyurethane foams at a range of temperatures, strain rates and initial densities.

For crushable and hysteretic foams, Faruque et al. (2) formulated a strain rate and temperature dependent constitutive model and implemented it in an explicit dynamic finite element code. The primary feature of the model is the inclusion of strain rate effect on Young’s modulus and yield strength, hardening of Young’s modulus with compaction and tension cutoff. The constitutive model is expressed in terms of modulus, which is a function of strain rate and volumetric strain (densification) as

$$E = E_0\left(1 + a \ln\left(\frac{\dot{\varepsilon}_{eff}}{\dot{\varepsilon}_0}\right)\right)\left(1 + b\left(\frac{\varepsilon_v}{\varepsilon_{0v}}\right)^2\right), \quad (6)$$

where $E_0$ is the quasistatic modulus at zero volume change, $a$ is the strain rate coefficient and $b$ is the densification coefficient, $\dot{\varepsilon}_{eff}$, $\varepsilon_0$, $\varepsilon_v$, $\varepsilon_{0v}$ are the effective strain rate, reference strain rate, volumetric strain and limiting volumetric strain beyond which the modulus is independent of volumetric strain, respectively.

For low-density polymeric foams subjected to high strain rate impact loading, Zhang et al. (3) proposed the following strain rate and temperature dependent constitutive equation and implemented it in a finite element code

$$\sigma(\varepsilon) = \sigma_0(T)\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^{a + b\varepsilon}. \quad (7)$$

Clearly, all the above models are phenomenological models in nature. A function such as the one defined in Eq 3 is meaningful in the interval $(0, \infty)$. The strain rate and temperature dependencies, either uncoupled as in Eq 4 or coupled as in Eq 5, play the role of scaling of the fundamental stress-strain curve. Equations 3, 4 and 7 may be used to better fit the experimental data, but the coupling of strain and strain rate in Eq 3 complicates the interpretation of the results. Equation 6 incorporates the strain rate into elastic modulus, but it is difficult to estimate the elastic modulus from high stress test data. Finally, all the above models are applicable for monotonic loading either in compression or tension but not in both. They are not suitable to describe the material behavior when the loading sequence changes from tension to compression or vice versa. Constitutive models that are applicable in both compressive and tensile regimes are of practical value because the majority of structures used in engineering applications are subjected to complex loading paths and a generalized form of stress-strain relationship is better than that for monotonic loading.

REQUIREMENTS FOR CONSTITUTIVE MODEL

In developing a generalized constitutive model applicable to foams under large deformations, the following relevant observations are noted from the available experimental data in the literature.

1. The shape of the stress-strain curve under compression is basically the same for almost all the crushable foams (2, 3, 8–21), and can be characterized by three different regimes as illustrated in Fig. 1: (i) a linear elastic regime OA or OA’; (ii) a plasticity-like stress plateau AB extending over a large range of strains where the foam starts to yield or crush progressively. Occasionally, a stress drop may be found at A’ before the stress reaches a plateau, which may be attributed to buckling of foam cell walls (20); and (iii) a densification regime BC where the stress rises sharply as the material starts to compact.
2. Unlike the compressive response, the tensile response "ODE" consists of a linear elastic regime followed by a progressive cell-wall stretch and breakage that may lead to some hardening-like behavior leading to fracture at E (2). Obviously, the strain to failure is significantly lower than that achieved in compression.

3. For the same loading rate and temperature, the higher the density of foam, the higher the stress level for a given strain, indicating that the characteristic curves spread out from low-density foams to high-density foams (22). For a given strain rate and density, the stress level decreases as the temperature increases (11, 12). It should be noted that the functional relationship between initial density, imposed strain rate, temperature and the stress is quite complicated and makes the task of deriving a unified analytical expression nontrivial.

Although a unified model encompassing all the above features is desirable, the current study focused on developing a stress-strain model applicable to a large group of foams with various initial densities and with the following capabilities.

- The model should capture the fundamental compressive and tensile stress-strain response in crushable foams of various initial densities subjected to large deformations.
- The function should be simple and continuously differentiable, i.e., no singularity in the entire domain. The number of parameters used must be as few as possible so as to facilitate interpretation of their meanings.
- The function should meet the initial condition, i.e., when strain is zero, the stress is also zero.
- The proposed stress-strain relationship should capture the effect of initial density. For example, (i) increased yield stress with increasing initial density, (ii) extended plateau region with higher initial porosity or lower initial density, etc. There are other features, such as simultaneous crushing and densification, that may occur in high-density foams and may give rise to a traditional hardening-like (gradually increasing) slope of the regime AB in Fig. 1. On the other hand, the slope of AB could be negative (because of buckling of cell walls) when the initial density of the foam is low (or the porosity is high), which gives rise to a softening effect (decreasing stress with increasing strain).

Our goal is to capture all the above features of the stress-strain behavior phenomenologically with a continuous function defined in the whole real domain.
CONSTITUTIVE MODEL FOR LARGE DEFORMATIONS

Functions of the form $\sigma = A\varepsilon^n$ defined in $(0, \infty)$ are well known for capturing the hardening effect of metal plasticity. Since the intent is to capture both the compressive and tensile responses, an exponential function that is continuously defined in $(-\infty, +\infty)$ is utilized. The proposed stress-strain relationship has the following form

$$\sigma = A \frac{e^{\alpha\varepsilon} + 1}{B + e^{\beta\varepsilon}}$$

(8)

where parameters $A$, $B$, $\alpha$ and $\beta$ are constant for a given initial density and strain rate, parameter $A$ has units of stress and other parameters do not have units. This function is continuously differentiable and passes through the origin of the stress-strain axes. The influences of the above parameters are shown in Fig. 2. The value of parameter $B$ is set to unity for simplicity. Parameter $A$ plays the role of scaling factor for yield stress and helps in determining the maximum (or the asymptote) if it exists. If $\alpha = \beta$, the function is monotonically increasing and is bounded between two asymptotes at $\sigma = \pm A$ yielding ideal plasticity-like response. Also note that the higher the magnitudes of $\alpha$ and $\beta$, the faster the function approaches the asymptotes. If $\alpha > \beta$, the function is monotonically increasing, but has only a lower bound at $\sigma = -A$.

The monotonic response can represent hardening-like behavior. The larger the difference between $\alpha$ and $\beta$, the steeper is the slope of hardening-like phase of the curve. If $\alpha < \beta$, the function is bounded and has two asymptotes at $\sigma = -A$ and $\sigma = 0$, and one theoretical maximum. This set of parameters can capture softening-like behavior. Again, the larger the difference between $\alpha$ and $\beta$, the steeper the softening-like phase of the curve. The parameters $\alpha$ and $\beta$ play the roles of squeezing or stretching the function.

The parameter $B$ plays a role of shifting the lower asymptote of the function as seen in Fig. 3 where all other parameters $A$, $\alpha$ and $\beta$ remain at unity. The lower asymptote can be analytically found as $\sigma = -A/B$ by allowing the exponential terms go to zero. Note that the stress-strain relationship is not strongly influenced in the interval of $(0, +\infty)$ when parameter $B$ is changed. It is seen from Fig. 2 that the function in Eq 8 is extremely sensitive to its parameters when the strain is close to zero or until yield stress is reached. Beyond this range, the asymptotes and/or the extremes are determined by the parameters $A$ and/or $B$ and the function is no longer sensitive to strain. In addition, $B$ in the denominator may be replaced by a constant, such as 1, or a simple function such that its flexibility is further enhanced to capture the tensile response. However, caution should be exercised not to choose a function to make the denominator $B(\varepsilon) + e^{\beta\varepsilon}$.
go to zero that leads to a singularity. Table 1 summarizes the above functional dependencies for various values of the parameters.

Note that the above function, proposed in Eq 8, cannot capture the rapid densification phase BC shown in Fig. 1. To capture this phenomenon, the following term is added to Eq 8,

$$\sigma = e^C(e^{\gamma e} - 1).$$

Therefore, the final function used to describe the entire stress-strain characteristics has the following form

$$\sigma = A\frac{e^{ue} - 1}{B + e^{ue}} + e^C(e^{\gamma e} - 1).$$

Table 1. Characteristics of the Proposed Stress-Strain Function.

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Asymptotes</th>
<th>Extreme</th>
<th>Monotonic</th>
<th>Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \beta$</td>
<td>Two, at $A$ and $\frac{-A}{B}$</td>
<td>Does not exist</td>
<td>Yes</td>
<td>Ideal plasticity-like</td>
</tr>
<tr>
<td>$\alpha &gt; \beta$</td>
<td>One, at $-\frac{A}{B}$</td>
<td>Does not exist</td>
<td>Yes</td>
<td>Hardening-like</td>
</tr>
<tr>
<td>$\alpha &lt; \beta$</td>
<td>Two, at $-\frac{A}{B}$ and $0$</td>
<td>One maximum</td>
<td>No</td>
<td>Softening-like</td>
</tr>
</tbody>
</table>

Fig. 3. Illustration of the influence of model parameter B, revealing a shift in the lower asymptote.
EXPERIMENTAL VALIDATION

Two types of foam with different initial bulk densities were used to validate the proposed constitutive model. These foams are heat-activated structural foams that are currently being used in automotive structures to improve strength and/or stiffness of structural members. The initial bulk densities of the foams were measured to be around 0.77 g/cm³ and 0.37 g/cm³. The porosity was measured using a Helium pycometer at 52% and 67%, respectively. These specimens have a length of 31.8 mm and a diameter of 18.5 mm. To obtain the complete quasistatic stress-strain characteristics, the specimens were tested under two conditions: uniaxial compression without any lateral confinement and uniaxial compression under rigid confinement. In the latter set of experiments, a specially designed steel confinement cell, shown in Fig. 5, was used to confine the specimens. Since the density and stiffness of the steel are an order of magnitude greater than those of foams, for all practical purposes the steel cell can be considered rigid. A thin polytetrafluoroethylene liner is used inside the steel cell to reduce friction between the foam specimen and the steel surface during the applied quasistatic loading. Since no lateral expansion is allowed in these experiments, the specimen can be subjected to large compressive deformations revealing the typical three regimes described in Fig. 1.

Figure 6 shows the two foam specimens before and after deformation under uniaxial compression without confinement, and Fig. 7 shows the specimens under uniaxial compression with rigid confinement. Clearly, specimens without confinement fail quickly along an inclined plane before accumulating large strain, whereas specimens under rigid confinement can be compressed to accumulate relatively large inelastic strains. Figure 8 illustrates the experimentally obtained stress-strain response for these two foams under uniaxial compression without lateral constraint. The high-density foam has a higher stiffness and higher yield stress. Beyond the elastic regime, the stress-strain response has a slightly negative slope where the foam cells start to crush locally in a narrow cross section of the specimen. The negative slope indicates that most of the cell crushing may occur via bucking of cell walls that leads to a gradual stress drop (softening). However, because the crushing process is confined to a narrow cross section, the specimen fails along a thin plane, as seen in Fig. 6, accumulating only a small strain of less than 10% before macroscopic fracture occurs.

Unlike the above response, the specimens under rigid confinement exhibit significantly larger strains extending well beyond 50% without undergoing any macroscopic failure, see Fig. 7. The deformation of the specimen is also more uniform because of the rigid
confinement and this leads to gradual densification indicated by the positive slope of the inelastic portion. Figure 9 illustrates the stress-strain response for the two foams. Note that the high-density foam has a higher yield stress, a smaller strain due to crushing of porous cells, and an earlier densification regime than those of the low-density foam. It is also clear that the low-density foam exhibits a more pronounced ideal plasticity-like plateau than the high-density foam during the crushing phase. This characteristic clearly stems from the fact that a low-density foam allows for easier crushing of cells, thus accumulating larger inelastic strains before densification starts to occur. Because the high-density foam has less room (compared to the low-density foam) for progressive crushing, this phase has a positive slope that combines crushing

confinement...
and simultaneous densification at a gradually increasing rate. The slope of this densification phase depends strongly on the initial density.

**MODELING THE STRESS-STRAIN BEHAVIOR**

The above features, i.e., increased yield stress with increasing initial density, extended plateau with increased initial porosity (or lower initial density) and early onset of densification with increased initial density, can all be captured by the parameters $A$ (scaling factor), $\alpha$ and $\beta$ (squeezing/extending factors) and $C$ and $\gamma$ (densification factors), respectively. To model the stress-strain curves under uniaxial compression shown in Fig. 8, the function defined in Eq 9 was utilized. The parameter sets for the two foams are given in Table 2. As can be seen in Fig. 8, the model captures...
the stress-strain response quite well. Note that the parameter $A$ clearly reflects the yield stress for the respective foams. In addition, the softening-like behavior beyond the yield stress was also fairly captured by the parameters $\alpha$ and $\beta$, where $\alpha < \beta$. The magnitude of the difference between these two parameters, i.e., $|\alpha - \beta|$, reflects the intensity of the slope of the curves beyond the yield stress. The larger the difference, the steeper the slope of the stress-strain curve (i.e., effect of low porosity). The proposed model captures these features effectively.

Unlike the above response, the rigidly confined specimens exhibit large accumulated strain (> 50%) with a rapid "densification" phase as seen in Fig. 9. Therefore, Eq 10 was used to model this response. The parameter sets for the two foams obtained from both partial and entire portion of experimental data are given in Table 3. Once again, it is seen that the function captures all the stress-strain features quite satisfactorily. The addition of the second term with parameters $C$ and $\gamma$ in Eq 10 allows for the rapid-densification phase to be modeled effectively. For the low-density foam that exhibits a more ideal plasticity-like stress plateau, the values of $\alpha$ and $\beta$ are almost the same. The slight upward trend is reflected in the slightly higher value of $\alpha$ compared to that of $\beta$. For the high-density foam that exhibits hardening-like behavior, $\alpha$ is significantly greater than $\beta$. The parameter $A$ once again captures the respective yield stresses of both foams effectively. As mentioned earlier, the parameter $\gamma$ reflects the rapid increase in the stress due to densification with a higher value of $\gamma$ for the high-density foam.

For the two different initial density foams experimentally investigated, the proposed phenomenological model can capture the characteristics accurately. Since the roles of the parameters in the function are known a priori, if foams of various initial densities

![Fig. 9. Stress-strain response of two foams under uniaxial compression with rigid confinement and model fit from using partial and entire experimental data. Note that the model obtained from fitting the entire data can make a reasonable extrapolation or prediction of the behavior beyond the used strain range.](image)

<table>
<thead>
<tr>
<th>Foam type</th>
<th>$A$ (MPa)</th>
<th>$B$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High density</td>
<td>11.64</td>
<td>1</td>
<td>2.854</td>
<td>2.890</td>
</tr>
<tr>
<td>Low density</td>
<td>7.026</td>
<td>1</td>
<td>2.328</td>
<td>2.390</td>
</tr>
</tbody>
</table>

Table 2. Parameters for Foams Under Uniaxial Compression Without Confinement.
(porosities) are available, the parameters can be expressed as functions of density or porosity based on the experimental data. If the foam behaviors under various strain rates, temperatures and confinement conditions are similar, the effects of strain rate and temperature can also be incorporated into the model by expressing the parameters as functions of these variables.

It is interesting to note that the model can also provide an equivalent elastic modulus for each density of foam. This modulus can be defined as

$$\tilde{E} = \lim_{\varepsilon \to 0} \frac{\partial \sigma}{\partial \varepsilon}$$

(11)

For the three-parameter model (Eq 9) in which B is set to unity, the equivalent modulus is given by

$$\tilde{E} = \lim_{\varepsilon \to 0} \frac{\partial \sigma}{\partial \varepsilon} = \lim_{\varepsilon \to 0} \left( \frac{\partial}{\partial \varepsilon} \left( A \frac{e^{\alpha \varepsilon} - 1}{1 + e^{\alpha \varepsilon}} \right) \right) = \frac{A \alpha}{2}$$

(12)

Clearly, this modulus varies with parameter A and α but is independent of β. However, as is seen in Table 3, both α and β have almost same numerical values and therefore, the equivalent modulus does not change significantly even if β is used in place of α in the above equation. Even for the five-parameter model (Eq 10), the equivalent elastic modulus is still dominated by the above parameters.

**DETERMINATION OF PARAMETERS**

It should be cautioned that the task of determining the parameters that describe the entire experimental data is not trivial. Since the proposed function is highly nonlinear, a note on the determination of suitable parameters is included. In this work, the nonlinear fitting function (nlinfit) in MATLAB® was used to fit the experimental data. This function employs a Gauss-Newton algorithm to conduct the iterative process where the parameters are refined in each step by minimizing the mean squared error between the given data set and the prediction of the nonlinear function. This algorithm is sensitive to the set of chosen initial values, a convergent set of parameters may not be reached if Eq 10 is used directly to fit the entire experimental data at once. The method adopted here includes three steps. First, only a portion of the data prior to rapid densification phase is used to obtain a convergent parameter set (A, B, α and β, B is set at 1 for simplicity) by utilizing the function defined in Eq 8. This step is relatively easy, because parameter A can be easily estimated from the yield stress and α and β can be estimated from the stress-strain characteristics (Table 1). The values of α and β may be slightly changed if a different portion of the data is selected. Second, the function in Eq 9 is added to the above equation where the value of the parameters A, (B = 1), α and β are known and the values for C and γ can be found. However, because the predetermined parameters A, (B) α and β are kept constant and only the other two parameters are adjusted in this step, the overall mean squared error has not been minimized.

Therefore, in the third step, all the above predetermined parameters are used as the initial values to obtain a final set of convergent parameters for the function defined in Eq 10. Thus, for a given set of experimental data, the final parameters can be reached irrespective of the portion of data used in the first step. However, if a different portion of the data is used in the second step, the parameters finally reached will also be different (see Table 3 and Fig. 9). A set of parameters obtained from the entire portion can best fit the experimental data, and can also predict the response beyond the used strain range. Note that the parameter set obtained from partial experimental data does not perform effectively since prediction or extrapolation beyond the used strain range may not have much credibility. It is recommended that the most effective way to capture the entire response is to use the entire experimental data set obtained by subjecting the specimen to the lowest possible porosity (or the highest density) level.

Note that the parameters of the model can be expressed as functions of initial porosity or bulk density if a large pool of experimental data can be generated at various initial porosity levels or bulk densities of foams. Similarly, each of the above parameters can also be functions of strain rate and applied temperature. Finally, in the Eqs 9 and 10, the stress ‘σ’ is strictly the uniaxial stress (σ11) along the loading direction. However, in the continued uniaxial-strain experiments, the lateral stress components (σ22 = σ33) have not yet been determined. Extension of the model to three dimensions obviously involves the Poisson’s effect and is a nontrivial job because of the complex behavior of porous materials. Currently, an active research program is investigating the above two issues.
CONCLUSION

A multi-parameter phenomenological model that captures the entire nonlinear stress-strain characteristics of structural porous materials under large deformations is proposed. Its effectiveness was demonstrated by the data obtained from two types of polymer foams of different initial density. The constitutive model employs at most six parameters and is versatile enough to capture the various physical characteristics of the macroscopic stress-strain response effectively. Five parameters are shown to adequately capture the compressive stress-strain behavior of crushable polymer foams under large deformation. The variation of yield stress (due to different initial density) can be captured by the scaling parameter $A$. The hardening-like or ideal plasticity-like or softening-like inelastic features can all be described by setting either $\alpha > \beta$, $\alpha = \beta$, or $\alpha < \beta$, respectively. The densification phase can be captured by the parameters $C$ and $\gamma$. It is proposed that further experimental study is needed to obtain the parameters as functions of initial material density, strain rate, and temperature.

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