A two-sample Anderson–Darling rank statistic

BY A. N. PETTITT

Department of Mathematics, University of Technology, Loughborough, Leicestershire

SUMMARY

A two-sample Anderson–Darling statistic is introduced and small-sample percentage points are given. An approximation to the distribution is also given. The statistic is related to Wilcoxon’s and Mood’s rank statistics. Asymptotic power comparisons are made with other two-sample rank statistics for shifts in location and scale.

Some key words: Anderson–Darling statistic; Asymptotic power; Distance statistic; Two-sample rank test.

1. INTRODUCTION

Anderson & Darling (1954) introduce the goodness-of-fit statistic

\[ A_n^2 = n \int_{-\infty}^{\infty} \frac{(F_n(x) - F_0(x))^2}{(1 - F_0(x))^2} dF_0(x) \]

to test the hypothesis that a random sample \( X_1, \ldots, X_n \), with sample distribution function \( F_n(x) \), comes from a continuous population with distribution function \( F_0(x) \). The function \( F_n(x) \) is defined as the proportion of the random sample \( X_1, \ldots, X_n \) which is not greater than \( x \). If there are two random samples \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \), and it is wished to test the null hypothesis that the samples come from identical continuous populations, then a two-sample extension of \( A_n^2 \) is

\[ A_{nm}^2 = \frac{nm}{N} \int_{-\infty}^{\infty} \frac{(F_n(x) - G_m(x))^2}{(1 - H_N(x))^2} dH_N(x), \] (1.1)

where \( F_n(x), G_m(x) \) and \( H_N(x) \), with \( N = n + m \), are the sample distribution functions of the \( X \)-sample, the \( Y \)-sample and the combined sample respectively, that is

\[ H_N(x) = \{nF_n(x) + mG_m(x)\}/N. \]

However, the integrand in (1.1) does not exist at the largest observation in the combined sample; consequently for \( A_{nm}^2 \) to exist the integrand is defined to be zero at the largest observation. Then \( A_{nm}^2 \) simplifies to

\[ \frac{1}{mn} \sum_{i=1}^{N-i} \frac{(M_i N - ni)^2}{i(N-i)}, \] (1.2)

where \( M_i \) is defined as the number of \( X \)'s less than or equal to the \( i \)th smallest in the combined sample, that is \( M_i = nF_n \circ H_N^{-1}(i/N) \), where \( H_N^{-1}(t) = \inf\{x: H_N(x) = t\} \).

It is easy to show that \( A_{nm}^2 \) has mean equal to 1, for \( E[(M_i N - ni)^2] = nm(i(N-i))/(N-1) \). This is the same mean as \( A_n^2 \). It seems impossible to find higher moments explicitly.
Table 1. Anderson–Darling two-sample statistic $A^2_{mn}$

Row 1: nominal critical $A_m$, so that $\Pr(A^2_{mn} \geq A_m)$ is nearest to $\alpha$. Row 2: $\Pr(A^2_{mn} \geq A_m)$. Row 3: large-sample approximation to distribution of $A^2_{mn}$, $\Pr(A^2_{mn} \geq (x_m - 1)(1-\frac{1}{55N}) + 1)$, where $\Pr(A^2 \geq x_m) = \alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$m$</th>
<th>$n$</th>
<th>$A^2_{mn}$</th>
<th>$A_m$</th>
<th>$A_m$</th>
<th>$A_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.02</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.03</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.04</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.05</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.06</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.07</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.08</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.09</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
<tr>
<td>0.10</td>
<td>0.05</td>
<td>0.10</td>
<td>2.586</td>
<td>3.564</td>
<td>3.564</td>
<td>3.564</td>
</tr>
</tbody>
</table>

Note: All use subject to JSTOR Terms and Conditions
The distribution of $A_{nm}^2$ was found by enumeration of all possible values of $A_{nm}^2$ by considering the $N!/(n!m!)$ orderings of $n$ $X$’s and $m$ $Y$’s. Use was made of an algorithm of Liu & Tang (1973). The upper percentage points of $A_{nm}^2$ are given in Table 1.

We also have found an approximation for the large-sample distribution of $A_{nm}^2$. It can be shown that $A_{nm}^2$ converges to the same limiting distribution as $A^2$ as the sample size increases. We denote by $A^2$ a statistic with the limiting distribution of $A_{nm}^2$. The distribution of $A^2$, and a few percentage points, are given by Anderson & Darling (1954). We can approximate reasonably well to the variance of $A_{nm}^2$ by $\frac{3}{5}(1 - 3.1/N)(\pi^2 - 9)$, where

$$\text{var}(A^2) = \frac{3}{5}(\pi^2 - 9).$$

We then find that the distribution of $+A_{nm}^2 = (A_{nm}^2 - 1)(1 + 1.55/N) + 1$ is well approximated by the distribution of $A^2$ in the upper tail. We have

$$E(+A_{nm}^2) = E(A^2), \quad \text{var}(+A_{nm}^2) = \text{var}(A^2).$$

From Table 1, it can be seen that this approximation is reasonably good, even where $m$ is much smaller than $n$.

Thus in large samples the percentage points of $A_{nm}^2$ can be approximated by

$$(x_a - 1)(1 - 1.55/N) + 1,$$

where $\text{pr}(A^2 \geq x_a) = \alpha$.

2. Relationship of $A_{nm}^2$ with other rank statistics

We can express $A_{nm}^2$ in the form

$$A_{nm}^2 = \frac{(N - 1)}{m} \int_0^1 y_n^2(t)/((t - t^2))d\nu_N, \quad (2.1)$$

where

$$y_n(t) = \sqrt{n} \{F_n \circ H_N^{-1}(t) - t\} \quad (2.2)$$

and $\nu_N$ is a measure that puts $1/(N - 1)$ at $i/N$ ($i = 1, \ldots, N - 1$). We could consider the continuous version of $A_{nm}^2$, that is

$$\frac{N}{m} \int_0^1 y_n^2(t)/((t - t^2))dt;$$

however, with probability $n/N$, this integral will not converge. This is because, when the largest observation is an $X$, then for $(N - 1)/N < t < 1$, $y_n(t) = \sqrt{n}\{(n - 1)/n - t\}$, and the integral diverges. Instead of (2.1) and (2.2), consider

$$\frac{N}{m} \int_0^1 x_n^2(t)/((t - t^2))d\nu_N, \quad (2.3)$$

where

$$x_n(t) = \begin{cases} \sqrt{n} \{F_n \circ H_N^{-1}(N + 1) - t\} & (0 \leq t \leq \frac{N}{N + 1}), \\ \sqrt{n}(1 - t) & \left(\frac{N}{N + 1} < t \leq 1\right). \end{cases}$$
and \( u_N \) is a measure that puts \( 1/N \) at \( i/(N+1) \) \( (i = 1, \ldots, N) \). The statistic given by (2.3) can be simplified to

\[
(\frac{mn}{n})^{-1} \sum_{i=1}^{N} \{ ((N+1)M_i - ni)^2 / [(N+1-i)] \}.
\]

The process \( \{x_n(t)\} \) is equivalent to the empirical process for a random sample consisting of the \( n \) values \( R_1/(N+1), R_2/(N+1), \ldots, R_n/(N+1) \), where \( R_i \) is the rank of \( X_i \) in the combined random sample. With this definition of \( x_n(t) \), the continuous version of (2.3) exists and is given by

\[
* A_{nm}^2 = (N/m) \int_0^1 x_n^2(t)/(t-t^2) \, dt
\]

\[
= - \left( \frac{N}{mn} \right) \left[ n^2 - 2n^3 \log (N+1) + \sum_{i=1}^{n} (2i-1) \{ \log R_i + \log (N+1-R_{N+1-i}) \} \right]
\]

(2.4)

simplifying the one-sample formula given by Anderson & Darling (1954). This enables an analogy to be made between \( * A_{nm}^2 \) and the one-sample statistic \( A_2^2 \). Durbin & Knott (1972) showed how \( A_n \) can be partitioned into statistics which are related to the Neyman–Barton ‘smooth’ tests (Kendall & Stuart, 1973, §30.37). The method is to consider the expansion of \( x_n(t)/(t-t^2)^{1/2} \) in terms of associated Legendre functions:

\[
f_j(z) = \left( \frac{2j+1}{j(j+1)} \right) P_j(z),
\]

where \( P_j(z) = (1-z^2)^{1/2} dP_j(z)/dz \), with \( z = 2t-1 \) and \( P_j(z) \) is the \( j \)th Legendre polynomial of the first kind. It can be shown that

\[
x_n(t)(t-t^2)^{-1/2} = \sum_{j=1}^{\infty} B_j \left( \frac{2j+1}{j(j+1)} \right) P_j(2t-1),
\]

where

\[
B_j = \left( j(j+1) \right)^{1/2} \int_0^{1} x_n(t)(t-t^2)^{-1/2} P_j(2t-1) \, dt
\]

\[
= - (2j+1)^{1/2} n^{-1} \sum_{i=1}^{N} P_j \left( \frac{2i}{N+1} - 1 \right) Z_i,
\]

on integrating by parts and noting that \( d[F_n \circ H_{N}^{-1} \{ (N+1)t/N \}] = Z_i/n \) at \( t = i/(N+1) \) and is 0 elsewhere. Here \( Z_i \) is the usual indicator function for the combined ordered sample, being 1 when the corresponding observation is an \( X \).

The statistic \( * A_{nm}^2 \) can be written in terms of the \( B_j \)'s:

\[
* A_{nm}^2 = \frac{N}{m} \sum_{j=1}^{\infty} \left( \frac{B_j^2}{j^2+j} \right). \tag{2.5}
\]

The \( \{B_j\} \) are just linear rank statistics, that is linear functions of the \( Z_i \) only. For example, with \( P_1(2t-1) = 2t-1 \),

\[
\left( \frac{N}{m} \right)^{1/2} B_1 = - \left( \frac{3N}{nm} \right)^{1/2} \sum_{i=1}^{N} \left( \frac{2i}{N+1} - 1 \right) Z_i
\]

\[
= - \left( \frac{N}{N+1} \right)^{1/2} \left[ \left( \frac{12}{mn(N+1)} \right)^{1/2} W - \left( \frac{3n(N+1)}{m} \right)^{1/2} \right],
\]

where \( W \) is the Wilcoxon statistic.
where \( W = \Sigma_i i Z_i \) is the Wilcoxon rank statistic. The moments of \( W \) are \( E(W) = \frac{1}{2} n(N+1) \) and \( \text{var}(W) = \frac{1}{12} mn(N+1) \), so that \( \{(N+1)/m\}^1/2 B_1 \) is the standardized form of the Wilcoxon statistic.

Also \( P_2(2t-1) = \frac{1}{2} \{3(2t-1)^2 - 1\} \), so that

\[
\left( \frac{N+1}{m} \right)^{1/2} B_2 = \left( \frac{180 m}{mn(N+1)^3} \right)^{1/2} M - \frac{1}{2} n(N+1)^2,
\]

where \( M = \Sigma \{i - \frac{1}{2}(N+1)\}^2 Z_i \) is Mood’s statistic. Now \( E(M) = \frac{1}{2} n(N^2 - 1) \) and

\[
\text{var}(M) = mn(N+1)(N^2 - 4)/180,
\]

so that in large samples \( \{(N+1)/m\}^1/2 B_2 \) is equivalent to the standardized form of Mood’s statistic.

The statistic \( A_{nm}^2 \) is related to linear rank statistics by (2.5), but in large samples \( A_{nm}^2 \) is equivalent to \( A_{nm}^2 \). Because of the simple formula, (1.2), for calculating \( A_{nm}^2 \), it seems reasonable to consider \( A_{nm}^2 \) rather than \( A_{nm}^2 \) in small samples.

The statistics \( \{B_{ij}\} \) are related to rank statistics found by another approach. Consider the sequence of values \( \{Z_1, \ldots, Z_N\} \) which we can express in terms of the orthogonal polynomials. Consider the standard orthogonal polynomials \( \{\phi_k(x)\} \) for equally spaced values of \( x \). If we imagine \( \{Z_1, \ldots, Z_N\} \) to take values at \( t_i = i/(N+1) \), so that \( t_i \) corresponds to \( x_i = i - \frac{1}{2}(N+1) \), then the function \( Z(t) \), taking values at \( t_i = i/(N+1) \) \( (i = 1, \ldots, N) \), can be expressed in terms of \( \{\phi_k(t)\} \) for \( k = 0, \ldots, N - 1 \).

If we have, for \( x_i = (N+1)t_i - \frac{1}{2}(N+1) \),

\[
Z(t_i) = \sum_{k=0}^{N-1} C_k \phi_k(x_i),
\]

then

\[
C_k = \sum_{i=1}^{N} Z_i \phi_k(x_i) \quad (k = 0, \ldots, N - 1).
\]

The \( \{C_k\} \) are just linear rank statistics, \( C_k = \Sigma \phi_k(i - \frac{1}{2}(N+1))Z_i \).

For example, \( \phi_1(x) = \lambda_1 x \), so that \( C_1 = \lambda_1 \{W - \frac{1}{2} n(N+1)\} \), \( W \) is Wilcoxon’s statistic, and \( \phi_2(x) = \lambda_2 \{x^2 - \frac{1}{2} N(N^2 - 1)\} \), so that \( C_2 = \lambda_2 \{M - \frac{1}{2} n(N^2 - 1)\} \), where \( M \) is Mood’s statistic.

If \( \lambda_k \) is chosen so that \( \Sigma \phi_k^2(x_i) = N(N-1)/(mn) \), then it is easy to show that the statistics \( \{C_k\} \) are uncorrelated rank statistics with mean zero and unit variance. Asymptotically \( \{(N+1)/m\}^1/2 B_2 \) is equivalent to \( C_k \).

### 3. Asymptotic Distribution and Power of the Statistic

#### 3.1. Asymptotic distribution of \( A_{nm}^2 \)

It is well known that \( (1 - \lambda)^{-2} y_n(t) \), where \( \lambda = n/N \), converges in distribution to a continuous normal process \( y(t) \) with mean 0 and covariance \( \min(s,t) - st \ (0 \leq s, t \leq 1) \); see for example Pyke & Shorack (1968). We have to show that in distribution

\[
A_{nm}^2 \rightarrow A^2 = \int_0^1 y^2(t)(t-t^2)^{-1} dt.
\]

Durbin (1973, § 4.4) indicates the difficulty of dealing with the asymptotic distribution of \( A_n^2 \). By the methods of Pyke & Shorack (1968), for \( \delta > 0 \), one can show

\[
(N/m) \int_0^{1-\delta} y_n^2(t)(t-t^2)^{-1} dt \rightarrow \int_0^{1-\delta} y^2(t)(t-t^2)^{-1} dt.
\]
If \( u = 1/\delta \) and \( u > 2 \), then using Theorem 4·2 of Billingsley (1968) it is necessary only to show

\[
\lim_{u \to \infty} \lim_{n \to \infty} \sup_{n} \Pr \left( \left| A_{nm}^{u} - (N/m) \int_{1/u}^{1-1/u} g_{n}^{u}(t) (t - t^{2})^{-1} \, dv_{N} \right| \geq \varepsilon \right) = 0
\]  

(3·1)

for each positive \( \varepsilon \), to prove \( A_{nm}^{u} \to A^{2} \) in distribution.

Assuming \( u < N \), otherwise (3·1) is trivially true, and denoting by \( B(u, n, m) \) the term in modulus signs in (3·1), then we have \( E[B(u, n, m)] = 2k/(N - 1) \), with \( k \) equal to the greatest integer not greater than \( N/u \); it can be shown that \( \text{var} \{ B(u, n, m) \} \leq K/u^{2} \), with \( K \) a fixed constant not dependent on \( u, n \) or \( m \).

Then, by using Chebychev’s Theorem, \( \Pr \{ B(u, n, m) \geq \varepsilon \} \leq K/(ue - 4)^{2} \) for all \( n, m \) and \( \varepsilon > 0 \) provided \( u > 4/\varepsilon \). Thus

\[
\lim_{n \to \infty} \sup_{n} \Pr \{ B(u, n, m) \geq \varepsilon \} \leq K/(ue - 4)^{2},
\]

and so as \( u, n \to \infty \), \( \lim \sup_{n} \Pr \{ B(u, n, m) \geq \varepsilon \} \to 0 \) for every fixed \( \varepsilon > 0 \). Hence \( A_{nm}^{u} \to A^{2} \) in distribution.

3·2. Asymptotic power of the statistic

If we consider an alternative hypothesis where the \( X \)-sample comes from a population with distribution function \( F(x) \), and the \( Y \)-sample from a population with distribution function \( G(x) \), with \( F(x) = F(x, \theta_{0}) \) and \( G(x) = F(x, \theta_{1}) \), where \( \theta \) is a vector parameter, then provided \( \theta_{1} = \theta_{0} - \gamma(\lambda(1 - \lambda) N)^{-1} \), where \( \gamma \) is a constant vector, \( (1 - \lambda)^{-1} y_{n}(t) \) converges to a normal process with mean \( \gamma' g(t) \) and covariance \( \min(s, t) - st \). The function \( g(t) \) is defined by

\[
g(t) = [\partial F/\partial \theta]_{\theta = \theta_{0}} \text{ and } t = F(x, \theta_{0}).
\]

This is a straightforward extension of the one-sample case considered by Durbin & Knott (1972). Now it can be shown that the asymptotic distribution of \( A_{nm}^{u} \), on this alternative, is given by \( \Sigma Z_{j}^{2}(\gamma^{2} + j) \), where \( \{ Z_{j} \} \) are independent normal random variables with mean \( \gamma^{j} \delta_{j} \) and the vector \( \delta_{j} \) is defined by

\[
\delta_{j} = (j^{2} + j)^{1/2} \int_{0}^{1} g(t) P_{j}^{1/2}(2t - 1)/(t - t^{2})^{-1/2} \, dt,
\]

and where \( P_{j}^{1/2} \) is the associated Legendre function defined in §2.

We consider, in particular, the null hypothesis that the \( X \) and \( Y \) populations are identical normal populations against the alternative of a shift in mean and variance of one of the populations. This is equivalent to taking \( F(x, \theta_{0}) \) to be \( N(0, 1) \) and \( F(x, \theta_{1}) \) to be

\[
N[ - \gamma_{1}(\lambda(1 - \lambda) N)^{-1}, 1 - \gamma_{2}(\lambda(1 - \lambda) N)^{-1}] .
\]

Stephens (1974) has already investigated the asymptotic power of the one-sample statistic \( A_{n}^{2} \) for a \( N(0, 1) \) versus \( N(\gamma_{1} n^{-1}, 1 + \gamma_{2} n^{-1}) \) test. The asymptotic distribution of \( A_{n}^{2} \) for this alternative can be shown to be the same as that of \( A_{nm}^{u} \) for the alternative for that statistic. Consequently we can use Stephens’s results. He finds values of \( (\gamma_{1}, \gamma_{2}) \) so that \( A_{n}^{2} \) has asymptotic power equal to 0·2, 0·4, 0·6 and 0·8 at the 0·05 significance level. Stephens also finds the asymptotic power of the Cramér–von Mises statistic \( W_{n}^{2} \), where

\[
W_{n}^{2} = \int_{-\infty}^{\infty} \{ F_{n}(x) - F_{0}(x) \}^{2} \, dF_{0}(x),
\]
A two-sample Anderson–Darling rank statistic

and of the Watson statistic \( U^2 \), for data on the circle, where

\[
U^2_n = n \int_{-\infty}^{\infty} \left[ F_n(x) - F_0(x) - \int_{-\infty}^{\infty} \{F_n(y) - F_0(y)\} \, dF_0(y) \right]^2 \, dF_0(x)
\]

for the same alternatives. Now \( W^2_n \) and \( U^2_n \) have their two-sample equivalents \( W^2_{nm} \) and \( U^2_{nm} \); see Anderson (1962) for \( W^2_{nm} \) and Burr (1964) for \( W^2_{nm} \) and \( U^2_{nm} \), or Durbin (1973, §6). The same remarks hold for \( W^2_{nm} \) and \( U^2_{nm} \) as do for \( A^2_{nm} \).

We can compare the asymptotic power of \( A^2_{nm} \) with a Wilcoxon–Bradley–Ansari statistic, \( T \), introduced by Lepage (1971), \( T = (W - \mu_W)/\sigma_W + (R - \mu_R)/\sigma_R \), where \( W \) is Wilcoxon’s rank statistic, and \( R \) is the Bradley–Ansari rank statistic for scale shift. Under the alternative considered above, it is easy to show that \( T \) is asymptotically \( \chi^2 \) with noncentrality parameter \( (3/\pi) \gamma_1^2 + (3/\pi^2) \gamma_2^2 \); since \( (W - \mu_W)/\sigma_W \) is asymptotically \( N\{-(3/\pi)^2 \gamma_1, 1\} \), \( (R - \mu_R)/\sigma_R \) is asymptotically \( N\{-(3/\pi)^2 \gamma_2, 1\} \), and \( W \) and \( R \) are uncorrelated. These results follow from considering the Pitman efficiencies of the tests.

We can also compare \( A^2_{nm} \) with a Wilcoxon–Mood statistic \( S \), the significance points of which have not been tabulated, where \( S = (W - \mu_W)/\sigma_W + (M - \mu_M)/\sigma_M \). Now \( S \) has an asymptotic \( \chi^2 \) distribution with noncentrality parameter \( (3/\pi) \gamma_1^2 + \{15/(4\pi^2)\} \gamma_2^2 \), since \( M \) is asymptotically \( N\{-15/(2\pi) \gamma_2, 1\} \) and \( W \) and \( M \) are uncorrelated, as seen above. It has already been shown that

\[
A^2_{nm} = \left( \frac{W - \mu_W}{\sigma_W} \right)^2 + \left( \frac{M - \mu_M}{\sigma_M} \right)^2 + \sum_{j=3}^{\infty} \frac{NB^2_j}{j(N - j)}
\]

so that \( A^2_{nm} \) is highly correlated with \( S \), and in large samples \( \text{corr}(A^2_{nm}, S) = 0.905 \).

The value of the noncentral parameter, \( d \), so that \( \text{pr}(C > 5.991) = 0.05 \) is \( d = 9.64 \), where \( C \) is \( \chi^2 \) with noncentrality parameter \( d \); 5.991 is the upper 0.05 significance point for central \( \chi^2 \). Thus the values of \( \{\gamma_1^2, \gamma_2^2\} \) so that \( T \) and \( S \) have asymptotic power equal to 0.05 at the 0.05 level are given by \( (3/\pi) \gamma_1^2 + (3/\pi^2) \gamma_2^2 = 9.64 \) and \( (3/\pi) \gamma_1^2 + \{15/(4\pi^2)\} \gamma_2^2 = 9.64 \), respectively.

The locus of points \( \{\gamma_1^2, \gamma_2^2\} \) with constant power in the \( \{\gamma_1^2, \gamma_2^2\} \) plane for either \( T \) or \( S \) is therefore a straight line. It is found from Stephens (1974) that the locus of points \( \{\gamma_1^2, \gamma_2^2\} \) so that \( A^2_{nm} \) has asymptotic power equal to 0.8 at the 0.05 level is very nearly a straight line. Similar comments hold for the loci for \( W^2_{nm} \) and \( U^2_{nm} \). These loci are drawn in Fig. 1. Also shown is the line \( \gamma_1^2 + (3/\pi) \gamma_2^2 = 9.64 \) which is the locus for the likelihood ratio test statistic, \( L \), for this test. Stephens calls these loci isodynes; the interpretation of an isodyne for a particular statistic at a given significance level and power is that the nearer the isodyne to the origin the better the test, since it requires less shift of \( \{\gamma_1^2, \gamma_2^2\} \) from the null hypothesis \( (0, 0) \) to achieve the required power. From Fig. 1, with \( A^2 \), \( W^2 \) and \( U^2 \) denoting the asymptotic statistics \( A^2_{nm}, W^2_{nm} \) and \( U^2_{nm} \) respectively, we see, as Stephens noted, \( A^2 \) is uniformly better than \( W^2 \), and \( A^2 \) is better than \( U^2 \), where the shift from the null is proportionately more in the location parameter, \( \gamma_1 \), than the variance parameter, \( \gamma_2 \). Further, \( A^2 \) is better than \( T \) or \( S \) when the shift is more in location than variance. Also, \( T \) is nearly always better than \( U^2 \) and \( S \) is uniformly better than \( T \).

The isodynes for different power levels are parallel for the \( \chi^2 \) statistics and approximately so for \( A^2 \). A change of significance level downwards to 1% affects \( A^2 \) by slightly shifting the isodyne away from the origin relative to the \( \chi^2 \) statistics; a slight shift the other way is found for an increase in significance level. That is, although we have chosen 0.8 power at the 5%
significance level, the relative positions of the isodynes are practically independent of power and significance level.

The author thanks a referee for some helpful comments, particularly on §3.

REFERENCES


[Received February 1975. Revised July 1975]
学霸图书馆
www.xuebalib.com

本文献由“学霸图书馆·文献云下载”收集自网络，仅供学习交流使用。

学霸图书馆（www.xuebalib.com）是一个“整合众多图书馆数据库资源，提供一站式文献检索和下载服务”的24小时在线不限IP图书馆。

图书馆致力于便利、促进学习与科研，提供最强文献下载服务。

图书馆导航：

图书馆首页 文献云下载 图书馆入口 外文数据库大全 疑难文献辅助工具