A primal-dual method for the Meyer model of cartoon and texture decomposition

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Funding information
NSFC, Grant/Award Number: 11871210; Construct Program of the Key Discipline in Hunan Province; Scientific Research Fund of Hunan Provincial Education Department, Grant/Award Number: 17A128; HKRGC GRF, Grant/Award Number: 1202715, 12306616, 12200317, 12300218, and HKBU RC-ICRS/16-17/03

Summary
In this paper, we study the original Meyer model of cartoon and texture decomposition in image processing. The model, which is a minimization problem, contains an \( l_1 \)-based TV-norm and an \( l_\infty \)-based \( G \)-norm. The main idea of this paper is to use the dual formulation to represent both TV-norm and \( G \)-norm. The resulting minimization problem of the Meyer model can be given as a minimax problem. A first-order primal-dual algorithm can be developed to compute the saddle point of the minimax problem. The convergence of the proposed algorithm is theoretically shown. Numerical results are presented to show that the original Meyer model can decompose better cartoon and texture components than the other testing methods.

KEYWORDS
\( G \)-norm, image decomposition, Meyer’s model, primal-dual algorithm, total variation

1 | INTRODUCTION

In many problems of image processing and computer vision, we have a target image that may contain different components such as cartoon component and oscillatory component. The cartoon component is the geometrical part or sketchy approximation of an image, which is well structured. It is used to model the homogeneous regions with sharp boundaries. The oscillatory component is a pattern of small-scale details and of zero mean. An important image processing task is to decompose the target image into these two components in order to improve the subsequent processing operations such as image denoising,\(^1\) image compression,\(^2\) image segmentation,\(^3,4\) face recognition,\(^5\) and image inpainting.\(^6\) For example, Meyer et al.\(^7\) proposed an image compression scheme where an image is treated as a superposition of cartoon component and texture component. Each components can be sparsely represented using a suitable dictionary, which is chosen such that each leads to a sparse representation over the content part it is serving, while yielding a nonsparse representation for the other content part. Examples of dictionaries that can sparsely approximate cartoon components in images are curvelets,\(^8,9\) orthonormal wavelets,\(^10\) and wavelet tight frames constructed by the unitary extension principle in the work of Ron et al.\(^11\) Examples of dictionaries that can sparsely approximate textures are the local discrete cosine transform and
the Gabor transform; see other works.\textsuperscript{12–16} If the cartoon component and the texture component are encoded separately, higher overall compression gain can be obtained.

The problem of image decomposition can be modeled to decompose a target gray image $f : (x, y) \in \Omega \to \mathbb{R}$ into two components

$$f = u + v,$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with a Lipschitz boundary, $u$ represents a cartoon component, and $v$ is the oscillatory component. Mathematically, the cartoon component can be described by a piecewise smooth function, which can be represented in the space of function of bounded variation

$$\text{BV}(\Omega) = \left\{ u \in L^1(\Omega) : \int_{\Omega} |Du| < \infty \right\},$$

where $Du$ denotes the generalized derivative of $u$ and $\int_{\Omega}|Du|$ stands for the total variation (TV) norm of $u$. When $u$ is regular, $\int_{\Omega}|Du|$ is often written as $\int_{\Omega}|\nabla u|$, which can be also reformulated as a dual approach as follows:

$$\int_{\Omega} |\nabla u| = \sup \left\{ \int_{\Omega} u \text{div} p : p = (p_1, p_2) \in Y, \|p\|_{\infty} \leq 1 \right\}.$$ 

Here, $\|p\|_{\infty} = \max_{x,y} |p(x,y)|$ with $|p(x,y)| = \sqrt{(p_1(x,y))^2 + (p_2(x,y))^2}$, $Y = L^2(\Omega) \times L^2(\Omega)$, and $\text{div} = -\nabla^* w$ is the adjoint operator $*$ of $w$. We denote the TV-norm of $u$ as $TV(u)$.

The oscillatory part $v$ contains essentially the noise and the texture. The noise is the unwanted component of the image acquired during the image acquisition process or the transmission process. Rudin, Osher, and Fatemi (ROF\textsuperscript{1}) proposed a TV minimization model to remove the Gaussian white noise

$$\min_{u + v = f} \frac{\beta}{2} \int_{\Omega} |v|^2 + TV(u),$$

where $\beta$ is a regularization parameter. The $L^2$-norm in the first term of (2) is replaced by $L^1$-norm for random noise or impulsive noise\textsuperscript{17} and by Kullback–Leibler (KL) divergence (I-divergence) for Poisson noise.\textsuperscript{18} Meyer characterized the solution $v$ in (2) in $G$ space,\textsuperscript{15} a dual space of $\text{BV}(\Omega)$, which is defined by

$$G(\Omega) = \{ v \in L^2(\Omega) : v = \text{div} w, w \in Y \}.$$ 

An image $v$ belongs to $G$-space if and only if there exists $w \in Y$ with $v = \text{div} w$. The $G$-norm of $v$ is defined as the infimum of all $L^\infty$-norm of the functions $w \in Y$ with $v = \text{div} w$, that is,

$$\|v\|_G = \inf \{ \|w\|_\infty : v = \text{div} w, w \in Y \}.$$ 

Meyer observed that the target image $f$ will be treated by the model (2) as texture or noise even though $f$ is regular (e.g., $f$ is a characteristic function) when the $G$-norm of $f$ is not larger than $1/\beta$. Theoretical analysis shows that a sequence of oscillating functions having bounded $L^q$-norm with $q \geq 2$ tends to zero in $G$-space without converging strongly to zero in $L^q(\Omega)$,\textsuperscript{19} Thus, if $v$ is the texture or oscillating patterns of the image $f$, it has a smaller value of $G$-norm than that of $L^2$-norm. Numerical experiments have shown that $G$-norm indeed captures oscillatory patterns and texture better than the standard ROF model.\textsuperscript{20,21} To better capture textures or oscillating patterns, Meyer proposed to replace the $L^2$-norm in the model (2) with $G$-norm\textsuperscript{15}; thus, the minimization model became

$$\min_{u + v = f} \beta \|v\|_G + TV(u).$$

Because $G$-norm cannot be expressed as integrals, the corresponding Euler–Lagrange equation of (3) cannot be expressed directly. It is a challenge to study numerical methods for solving the minimization problem in (3). In the literature, several numerical methods\textsuperscript{20–22} have been proposed to modify Meyer’s model (3) and use the other norms, which can be implemented numerically in practice, to approximate $G$-norm. Vese and Osher (VO\textsuperscript{21}) proposed to relax the constraint $f = u + v$ and approximate $G$-norm by $L^p$-norm in the original Meyer model (3). According to the definition of
G-norm, if \( \mathbf{v} \in G(\Omega) \), then there exists \( \mathbf{w} \in Y \) with \( \mathbf{v} = \text{div}\mathbf{w} \) such that \( \| \mathbf{v} \|_G = \| \mathbf{w} \|_\infty \) (see lemma 5.2.1 in the work of Aubert et al.\(^{19}\)). As a result, the minimization problem (3) can be rewritten as

\[
\min_{u, \mathbf{v}} \beta \| \mathbf{w} \|_\infty + \text{TV}(\mathbf{u}),
\]

subject to \( u + \mathbf{v} = f \) and \( \mathbf{v} = \text{div}\mathbf{w} \). To numerically compute the solution of (4), Vese et al.\(^{21}\) proposed to use \( L^p \)-norm to replace \( L^\infty \)-norm in (4); then, the minimization problem considered in the work of Vese et al.\(^{21}\) is

\[
\min_{u, \mathbf{v}} \frac{\alpha}{2} \int_{\Omega} |f - u - \text{div}\mathbf{w}|^2 + \beta \left( \int_{\Omega} |\mathbf{w}|^p \right)^{\frac{1}{p}} + \text{TV}(\mathbf{u}),
\]

where \( \alpha \) and \( \beta \) are tuning parameters. The first and second terms ensure \( \text{div}\mathbf{w} \approx f - u \in G(\Omega) \). If \( \alpha \to \infty \) and \( p \to \infty \), the VO model mathematically coincides with the original Meyer model in (4); see (3). Once the solution \( (\hat{u}, \hat{\mathbf{v}}) \) of (5) is obtained, the texture component is simply given by \( \hat{\mathbf{v}} = \text{div}\hat{\mathbf{w}} \). Formally, minimizing the objective function in (5) with respect to \( u, \mathbf{w} \) yields the Euler–Lagrange equations. Later, Osher et al.\(^{20}\) considered the case \( p = 2 \), which approximates \( L^\infty \)-norm by the seminorm in \( H^{-1} \).

Aujol, Aubert, Blanc-Feraud, and Chambolle\(^{22}\) (AABC) proposed to approximately solve the original Meyer model by replacing the \( G \)-norm in (3) with a constraint \( \| \mathbf{v} \|_G \leq \mu \) with a parameter \( \mu > 0 \) and adding extra regularization terms

\[
\min_{u, \mathbf{v}} \frac{\alpha}{2} \int_{\Omega} |f - u - \mathbf{v}|^2 + \text{TV}(\mathbf{u}), \quad \text{subject to } \| \mathbf{v} \|_G \leq \mu.
\]

Here, \( \alpha \) is a regularization parameter. When \( \alpha \to \infty \), the problem in (6) tends to the limit problem

\[
\min_{u, \mathbf{v}} \text{TV}(\mathbf{u}), \quad \text{subject to } \| \mathbf{v} \|_G \leq \mu, f = u + \mathbf{v}.
\]

Mathematically, (3) and (7) are equivalent. Assume that \( (\hat{u}, \hat{\mathbf{v}}) \) is the solution of (3), it will also be the solution of problem (7) for a particular parameter \( \mu = \| \hat{\mathbf{v}} \|_G \). Similarly, if there exists a solution for problem (7), it should be a solution of (3) for a particular choice of \( \beta \), which is the Lagrange multiplier corresponding to the constraint \( \| \mathbf{v} \|_G \leq \mu \) in (7). Thus, the problem (6) asymptotically converges to the original Meyer model (3) when \( \alpha \) tends to infinity. The solution of the AABC model (6) is unique\(^{22}\) and can be obtained by the alternating minimization method. Fixed the variable \( \mathbf{v} \), we obtain the \( u \)-subproblem, which is the classical TV denoising problem. While fixed the variable \( u \), we obtain the \( \mathbf{v} \)-subproblem, which is equivalent to compute the projection of \( f - u \) to the set \( \{ \mathbf{v} : \| \mathbf{v} \|_G \leq \mu \} \). The corresponding \( u \)- and \( \mathbf{v} \)-subproblems can be solved approximately by Chambolle’s projection method.\(^{23}\) Gilles et al. proposed to solve (6) by the splitting Bregman iteration method.\(^{24}\)

In the literature, other spaces were used to model the oscillating component. Aujol et al.\(^{25}\) proposed to replace \( G \)-norm in (6) by \( E \)-norm,\(^{15}\) which is the norm endowed in the \( E \)-space, the dual space of the standard Besov space \( B_1^1 \). The \( E \)-norm model can well represent noise and, hence, is suited to denoising a textured image. In the work of Aujol et al.\(^{26}\) a Hilbert-space norm was proposed to represent the oscillatory component. The norm of the oscillatory component can be easier to compute by convolution with a smoothing kernel in \( L^2 \)-norm. Elion et al.\(^{27}\) proposed to model the oscillatory part as the Laplacian of a single-valued function whose gradient belongs to \( L^\infty \) space. A splitting Bregman iteration approach was applied to find the solution.\(^{26}\) In the work of Ng et al.,\(^{29}\) the authors proposed a cartoon+texture decomposition model to restore blurred images with missing pixels.

The main goal of this paper is to develop a numerical scheme to solve the original Meyer model in (3) rather than an approximation one. To overcome the numerical difficulty caused by the \( G \)-norm, we reformulate it using Legendre–Fenchel duality to represent the \( G \)-norm

\[
\| \mathbf{v} \|_G = \sup_{\mathbf{z}} \left\{ \int_{\Omega} \mathbf{w} \cdot \mathbf{z}, \mathbf{w} \in Y, \mathbf{v} = \text{div}\mathbf{w}, \int_{\Omega} |\mathbf{z}| \leq 1 \right\}.
\]

Denote \( A = \{ p : p \in Y, \| p \|_\infty \leq 1 \} \) and \( B = \{ \mathbf{z} : \mathbf{z} \in Y, \int_{\Omega} |\mathbf{z}| \leq 1 \} \). Notice that if \( \mathbf{v} \in G(\Omega) \), then there exists \( \mathbf{w} \in Y \) with \( \mathbf{v} = \text{div}\mathbf{w} \). Once \( \mathbf{w} \) is given, the cartoon component of \( f \) is obtained by \( u = f - \text{div}\mathbf{w} \). We use the duality to represent both
We will show that the infimum and the supremum in (8) can be swapped and then apply the primal-dual proximal point method\textsuperscript{30–37} to compute the solution of the minimax problem (8) alternately with the primal variable \( \mathbf{w} \) and the dual variables \( \mathbf{p}, \mathbf{z} \) fixed alternatively.

We highlight the major differences between our method and the previous methods\textsuperscript{20–22,24} to solve the G-norm–based image decomposition problem. First of all, the existing methods\textsuperscript{20–22,24} were to solve a related problem, which was to approximate the original one (3). Through asymptotic analysis, it can be shown that the VO model (5) and the AABC model (6) mathematically coincide with the original Meyer model (3). However, for a fixed parameter \( \alpha \), they are of course different from the original model (3). To our best knowledge, no work is proposed in the literature to develop an algorithm of the proposed approach is that there are closed-form solutions for each subproblem in the minimization process. Our numerical results also show that the cartoon component obtained by the original Meyer model has a sharper boundary than the AABC model. Moreover, the advantage of the proposed approach is that there are closed-form solutions for each subproblem in the minimization process.

The rest of this paper is organized as follows. In Section 2, we study a primal-dual algorithm to solve the minimax problem (8) numerically. We also describe the algorithm using a discrete setting. In Section 3, we give the convergence result for the proposed algorithm. In Section 4, we present some numerical examples for image decompositions. In Section 5, a conclusion is finally given.

## 2 | PRIMAL-DUAL ALGORITHM

In this section, we consider to calculate the solution of the minimax problem (8) numerically. The presentation is given below in the discrete setting, but it is the same as in the continuous case.

### 2.1 | Notations

Assume that the size of the discrete image is \( n \times n \). We denote by \( X \) the Euclidean space \( \mathbb{R}^{n \times n} \), and \( Y = X \times X \). We use the classical notions \( \mathbf{f}_{i,j} = \mathbf{f}(ih, jh), \mathbf{u}_{i,j} = \mathbf{u}(ih, jh), \mathbf{v}_{i,j} = \mathbf{v}(ih, jh) \), and \( \mathbf{w}_{i,j} = (\mathbf{w}_{1,i,j}, \mathbf{w}_{2,i,j}) \) with \( \mathbf{w}_{r,i,j} = \mathbf{w}_{r}(ih, jh)(r = 1, 2) \).\( \mathbf{p}_{i,j} = (\mathbf{p}_{1,i,j}, \mathbf{p}_{2,i,j}) \) with \( \mathbf{p}_{r,i,j} = \mathbf{p}_{r}(ih, jh)(r = 1, 2) \). \( \mathbf{z}_{i,j} = (\mathbf{z}_{1,i,j}, \mathbf{z}_{2,i,j}) \) with \( \mathbf{z}_{r,i,j} = \mathbf{z}_{r}(ih, jh)(r = 1, 2) \), where \( h > 0 \) is the step size and \((ih, jh) \) denotes a discrete point for \( 1 \leq i, j \leq n \).

Define the inner product \( \langle \mathbf{u}, \mathbf{f} \rangle_X = \sum_{i,j} \mathbf{u}_{i,j} \mathbf{f}_{i,j} \), and the norm \( \| \mathbf{p} \|_2 = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle_X} \). Let \( \| \mathbf{p} \|_1 = \sum_{i,j} |\mathbf{p}_{i,j}| \), and \( \| \mathbf{p} \|_\infty = \max_{i,j} \{|\mathbf{p}_{i,j}|\} \). For any \( \mathbf{u} \in X \), the discrete gradient of \( \mathbf{u} \) is defined as

\[
(\nabla \mathbf{u})_{i,j} = \left( (\nabla_x \mathbf{u})_{i,j}, (\nabla_y \mathbf{u})_{i,j} \right).
\]

Here,

\[
(\nabla_x \mathbf{u})_{i,j} = \mathbf{u}_{i+1,j} - \mathbf{u}_{i,j} \quad \text{and} \quad (\nabla_y \mathbf{u})_{i,j} = \mathbf{u}_{i,j+1} - \mathbf{u}_{i,j}
\]

with \( \mathbf{u}_{i+1,j} = \mathbf{u}_{i,j} \) and \( \mathbf{u}_{i,j+1} = \mathbf{u}_{i,j} \) for \( i, j = 1, 2, \ldots, n \). The discrete version of the divergence operator is defined by analogy with the continuous case by \( \text{div} = -\nabla^T \). Thus, the discrete divergence of \( \mathbf{p} \) is defined as

\[
(\text{div} \mathbf{p})_{i,j} = \mathbf{p}_{1,i,j} - \mathbf{p}_{1,i-1,j} + \mathbf{p}_{2,i,j} - \mathbf{p}_{2,i,j-1}
\]

with \( \mathbf{p}_{1,i,0} = \mathbf{p}_{2,i,0} = 0 \) for \( i, j = 1, 2, \ldots, n \). Moreover, for every \( \mathbf{p} \in Y \) and \( \mathbf{f} \in X \), we have \( \langle -\text{div} \mathbf{p}, \mathbf{f} \rangle_X = \langle \mathbf{p}, \nabla \mathbf{f} \rangle_Y \).

### 2.2 | Numerical method

First, we describe the minimax problem in a discrete setting. The discrete version of (8) can be reformulated as

\[
\mathcal{J}(\mathbf{w}, \mathbf{p}, \mathbf{z}) = \langle \mathbf{f} - \text{div} \mathbf{w}, \text{div} \mathbf{p} \rangle_X + \beta \langle \mathbf{w}, \mathbf{z} \rangle_Y.
\]
The sets $A, B$ are defined as $A = \{ p \mid p \in Y, \| p \|_{\infty} \leq 1 \}$ and $B = \{ z \mid z \in Y, \| z \|_{1} \leq 1 \}$. It is clear that $A, B$ are convex and closed.

We now recall that a pair $(w^*, p^*, z^*)$ is a saddle point of $J(w, p, z)$ if

$$J(w^*, p^*, z^*) = \inf_{w} \sup_{p} J(w, p, z) = \inf_{z} \sup_{p} J(w, p, z).$$

We note that $J(\cdot, p, z)$ and $J(w, \cdot, \cdot)$ are linear functions for each $p, z$ and each $w$, respectively. It is well known that a linear function is both convex and concave.

We will demonstrate that the set of saddle points of $J$ defined in (9) is nonempty. Let

$$\psi(w) = \begin{cases} \sup_{p \in A, z \in B} J(w, p, z) = TV(f - \text{div} w) + \beta \| w \|_{\infty}, & \text{if } w \in \Omega, \\ +\infty, & \text{if } w \notin \Omega. \end{cases}$$

It is easier to know that $\psi(w) > -\infty$ for all $w$ and $\psi(0) < \infty$, which implies that $\psi(w)$ is proper. As $\{w : \psi(w) \leq \lambda\} \subseteq \{w : \|w\|_{\infty} \leq \lambda\}$, it implies that the level sets $\{w : \psi(w) \leq \lambda\}$ are closed and bounded, and they are compact. Let

$$\phi(p, z) = \begin{cases} -\inf_{w \in \Omega} J(w, p, z), & \text{if } p \in A, z \in B, \\ +\infty, & \text{if } p \notin A \text{ or } z \notin B. \end{cases}$$

We know that $\phi(p, z) = 0$ if $\nabla \text{div} p + \beta z = 0$ and $p \in A, z \in B$, and $\phi(p, z) = +\infty$ if $\nabla \text{div} p + \beta z \neq 0$. Therefore, the level sets $\{(p, z) : \phi(p, z) \leq \lambda\}$ are closed and bounded, and they are compact. Hence, according to the saddle-point theorem (proposition 5.5.6 in the work of Bertsekas$^{38}$), the set of the saddle points of $J(w, p, z)$ is nonempty.

By the existence of the saddle point of $J(w, p, z)$, convex analysis can be applied to show that the infimum and the supremum in (8) can be swapped, that is,

$$\inf_{w} \sup_{p \in A, z \in B} J(w, p, z) = \sup_{p \in A, z \in B} \inf_{w} J(w, p, z) = J(w^*, p^*, z^*).$$

Numerous methods have been proposed to solve saddle-point problems. They include the finite envelope method, the primal-dual gradient algorithm$^{39}$ and the alternating direction method (see the work of Bertsekas et al.$^{40}$ for discussion). Here, we apply the (primal-dual) proximal-point method$^{40}$ to compute the saddle point of (8). In particular, proximal-point iterations are applied to the subdifferential of the primal-dual function alternately with the primal variable and the dual variable fixed. This can be equivalently interpreted as a first-order primal-dual relaxed Arrow–Hurwicz algorithm, proposed in the work of Chambolle et al.$^{30}$ More specifically, starting from an initial guess $(w^{(0)}, p^{(0)}, z^{(0)})$, this method computes a sequence of iterates:

$$w^{(k+1)} = \arg \min_{w} J(w, p^{(k)}, z^{(k)}) + \frac{1}{\Delta s} \| w - w^{(k)} \|_{2}^{2}$$

$$+ \frac{1}{\Delta s} \| w - w^{(k)} \|_{2}^{2},$$

$$z^{(k+1)} = \arg \max_{z \in B} J(w^{(k+1)}, p, z) - \frac{1}{\Delta t} \| z - z^{(k)} \|_{2}^{2}.$$

$$p^{(k+1)} = \arg \max_{p \in A} J(w^{(k+1)}, p, z) - \frac{1}{\Delta t} \| p - p^{(k)} \|_{2}^{2}.$$
2.2.1 Subproblem for variable \( w \)

Because \( J(w, p^{(k)}, z^{(k)}) \) is linear with respect to \( w \), we see that the objective function in (10) is quadratic with respect to \( w \), that is, \( w^{(k+1)} \) is the minimizer of the quadratic function \( \varphi_k(w) \), which is defined by

\[
\varphi_k(w) \equiv \frac{1}{2s} \| w - w^{(k)} \|_2^2 + \langle w, \nabla \text{div} p^{(k)} + \beta z^{(k)} \rangle_y.
\]

Hence, \( w^{(k+1)} \) can easily be computed by the formula

\[
w^{(k+1)} = w^{(k)} - s \nabla \text{div} p^{(k)} - \beta s z^{(k)}.
\]

Because \( \varphi_k(w) \) is strongly convex functions with modulus \( 1/s \), it follows that (proposition 6 in the work of Rockafellar36)

\[
\varphi_k(w) \geq \varphi_k(\hat{w}) + \langle \nabla \varphi_k(\hat{w}), w - \hat{w} \rangle_Y + \frac{1}{2s} \| w - \hat{w} \|_2^2
\]

for any \( w \) and \( \hat{w} \). Notice that \( w^{(k+1)} = \arg\min_w \varphi_k(w) \), we have \( \nabla \varphi_k(w^{(k+1)}) = 0 \). Setting \( \hat{w} = w^{(k+1)} \) in the inequality (14), we obtain

\[
\| w - w^{(k)} \|_2^2 + 2s \langle w - w^{(k+1)}, \nabla \text{div} p^{(k)} + \beta z^{(k)} \rangle_Y \geq \| w^{(k+1)} - w^{(k)} \|_2^2 + \| w - w^{(k+1)} \|_2^2.
\]

(15)

2.2.2 Subproblem for variable \( p \)

We consider to solve the subproblem (12). Notice that

\[
p^{(k+1)} = \arg\min_{p \in A} \frac{1}{2} \| p - p^{(k)} \|_2^2 - t \langle f - \text{div} \overline{w}^{(k+1)}, \text{div} p \rangle_x
\]

\[
= \arg\min_{p \in A} \frac{1}{2} \| p - \left( p^{(k)} - t \nabla (f - \text{div} \overline{w}^{(k+1)}) \right) \|_2^2.
\]

Thus, the solution \( p^{(k+1)} \) is the projection of \( (p^{(k)} - t \nabla (f - \text{div} \overline{w}^{(k+1)})) \) onto the set \( A \). The projection of a vector \( q \) onto \( A \) can be conveniently expressed as

\[
P_A(q) = \arg\min_{p \in A} \| p - q \|_2^2.
\]

(16)

The projection operator \( P_A \) can be computed by the Lagrangian method. The Lagrangian function associated with (16) is

\[
\| p - q \|_2^2 + \sum_{i,j} \beta_{i,j} \left( |p_{i,j}|^2 - 1 \right),
\]

where \( \beta_{i,j} \geq 0 \) is the Lagrange multiplier associated with the constraint \( |p_{i,j}|^2 \leq 1 \). Its complementarity condition implies that, for the optimal \( \beta_{i,j} \), either \( \beta_{i,j} = 0 \) with \( |p_{i,j}| \leq 1 \) and \( |q_{i,j}| \leq 1 \), or \( \beta_{i,j} > 0 \) with \( |p_{i,j}| = 1 \) and \( |q_{i,j}| > 1 \). In the former case, we have \( p_{i,j} = q_{i,j} \). In the latter case, the Karush–Kuhn–Tucker conditions yield

\[
p_{i,j} - q_{i,j} + \beta_{i,j} p_{i,j} = 0, \quad \forall i, j.
\]

Therefore, we have \( \beta_{i,j} = |q_{i,j}| - 1 \), and thus, \( p_{i,j} = q_{i,j}/|q_{i,j}| \). Hence, we obtain

\[
(P_A(q))_{i,j} = \frac{q_{i,j}}{\max(1, |q_{i,j}|)}.
\]

Therefore, we have

\[
p^{(k+1)} = P_A \left( p^{(k)} - t \nabla (f - \text{div} \overline{w}^{(k+1)}) \right).
\]

2.2.3 Subproblem for variable \( z \)

The subproblem (13) can be reformulated as

\[
z^{(k+1)} = \arg\min_{z \in B} \frac{1}{2t} \| z - (z^{(k)} + \beta t \overline{w}^{(k+1)}) \|_2^2.
\]

Thus, \( z^{(k+1)} \) is the projection of \( (z^{(k)} + \beta t \overline{w}^{(k+1)}) \) onto the set \( B \). The projection of a vector \( q \) onto \( B \) can be conveniently expressed as

\[
P_B(q) = \arg\min_{p \in B} \| p - q \|_2^2.
\]

(17)
Hence, the subproblem (13)
\[ z^{(k+1)} = P_\beta(z^{(k)}) + \beta(\|w^{(k+1)}\|). \]

We also apply the Lagrangian method to calculate the projection operator \( P_\beta \). Instead of solving the problem (17), we consider the following unconstrained problem:
\[ \min_p \left\{ \frac{1}{2}\|p - q\|_2^2 + \beta(\|p\|_1 - 1) \right\}. \]

Here, \( \beta \) is the Lagrange multiplier corresponding to the inequality constraint \( \|p\|_1 \leq 1 \). Its complementarity condition implies that, for the optimal \( p \), either (i) \( \|p\|_1 \leq 1 \) with \( \beta = 0 \) or (ii) \( \|p\|_1 = 1 \) with \( \beta > 0 \).

For (i), we can deduce that \( \|q\|_1 \leq 1 \) and the solution is given by \( p = q \). Now, we consider how to find the solution for the case (ii). For (ii), there exists a suitable parameter \( \beta \) such that the solution of (18) is given by
\[ p_i = \begin{cases} q_i - \frac{\beta}{|q_i|}, & \text{if } |q_i| > \beta, \\ 0, & \text{otherwise}. \end{cases} \]

According to the complementary condition, the parameter \( \beta > 0 \) should satisfy
\[ \|p\|_1 = \sum_{i=1}^{L} \min\{1, |q_i| \} = \beta \sum_{i=1}^{L} |q_i| \left| 1 - \frac{\beta}{|q_i|} \right| = 1. \]

Notice that \( 0 < 1 - \beta/|q_i| < 1 \) for \( |q_i| > \beta \), the above inequality implies that
\[ 1 = \|p\|_1 \leq \sum_{i=1}^{L} |q_i| \leq \|q\|_1. \]

If \( \|q\|_1 > 1 \), we know that there exists \( \beta > 0 \).

Now, we consider how to find \( \beta \) to satisfy the Equation (20). Let \( r \) be the vector obtained by sorting the elements of the vector \( |q| \) in ascending order, that is,
\[ r_1 \leq r_2 \leq \cdots \leq r_L. \]

Assume that \( i_0 \) is the minimum index such that \( r_{i_0} \geq \beta \), the equation in (20) can be rewritten as
\[ \|p\|_1 = \sum_{j=i_0}^{L} (r_j - \beta). \]

We let \( r_0 = 0 \) and define the sequence \( E_i (i = 0, 1, \ldots, L - 1) \) as follows:
\[ E_i = \sum_{j=i+1}^{L} (r_j - r_i). \]

Then, \( E_i \) is a decreasing sequence with \( E_0 = \sum_{j=1}^{L} r_j = \|q\|_1 \geq 1 \). We have either \( E_{L-1} \geq 1 \) or \( E_{L-1} < 1 \). When \( E_{L-1} \geq 1 \), we have \( r_{L-1} \geq r_{L-1} + 1 \). We set \( \beta = r_{L-1} - 1 \). When \( E_{L-1} < 1 \), there must exist an index \( i_0 < L - 1 \) such that \( E_{i_0} \geq 1 > E_{i_0+1} \). We consider the function
\[ \varphi(x) = \sum_{j=i_0+1}^{L} (r_j - x). \]

We can verify that \( \varphi(r_{i_0}) = E_{i_0} \) and \( \varphi(r_{i_0+1}) = E_{i_0+1} \). The function \( \varphi(x) \) is continuous in \( [r_{i_0}, r_{i_0+1}] \) with \( \varphi(r_{i_0}) \geq 1 \) and \( \varphi(r_{i_0+1}) < 1 \); there must exist a \( \beta \) such that \( \varphi(\beta) = 1 \). Hence, we have
\[ \beta = r_{i_0} + \frac{E_{i_0} - 1}{L - i_0}. \]
3 | CONVERGENCE ANALYSIS

We show that the sequence \( \{ (w^{(k)}, p^{(k)}, z^{(k)}) \} \) generated by (10)-(13) converges to the saddle point of \( J(w, p, z) \) in (9).

First, we introduce the inner product of the vector \( x_i = \begin{pmatrix} w_i \\ p_i \\ z_i \end{pmatrix} \) \((i = 1, 2)\) with \( w_i, p_i, z_i \in Y \). We define the inner product

\[
\langle x_1, x_2 \rangle = \langle w_1, w_2 \rangle_Y + \langle p_1, p_2 \rangle_Y + \langle z_1, z_2 \rangle_Y.
\]

We have the following lemma.

**Lemma 1.** Assume that \( s, t, \theta \) are defined by (10)-(13). Let \( \lambda = \| \nabla \text{div} \|_2^2 \) and

\[
W = \begin{pmatrix}
\frac{1}{s}I & -\nabla \text{div} & -\beta I \\
-\nabla \text{div} & \frac{1}{t}I & 0 \\
-\beta I & 0 & \frac{1}{t}I
\end{pmatrix}.
\]

Then, when \( s(\lambda + \beta) < 1 \) and \( t < \min(1, 1/\beta) \), \( W \) is a positive definite matrix.

**Proof.** Denote \( x = \begin{pmatrix} w \\ p \\ z \end{pmatrix} \) with \( w, p, z \in Y \), we have

\[
\langle x, Wx \rangle = \left\langle w, \frac{1}{s}w - \nabla \text{div} p - \beta z \right\rangle_Y + \left\langle p, -\nabla \text{div} w + \frac{1}{t}p \right\rangle_Y + \left\langle z, -\beta w + \frac{1}{t}z \right\rangle_Y
\]

\[
= \frac{1}{s} \| w \|_2^2 + \frac{1}{t} \| p \|_2^2 + \frac{1}{t} \| z \|_2^2 - 2(\langle p, \nabla \text{div} w \rangle_Y - 2\beta(\langle w, z \rangle_Y).
\]

Using the inequality \( 2ab \leq a^2 + b^2 \), we obtain

\[
\langle x, Wx \rangle \geq \frac{1}{s} \| w \|_2^2 + \frac{1}{t} \| p \|_2^2 + \frac{1}{t} \| z \|_2^2 - \| \nabla \text{div} \|_2^2 \| w \|_2^2 - \beta \| w \|_2^2 - \| z \|_2^2
\]

\[
= \left( \frac{1}{s} - \| \nabla \text{div} \|_2^2 - \beta \right) \| w \|_2^2 + \left( \frac{1}{t} - 1 \right) \| z \|_2^2 + \left( \frac{1}{t} - \beta \right) \| z \|_2^2.
\]

Therefore, when \( s(\lambda + \beta) < 1 \) and \( t < \min(1, 1/\beta) \), we have \( \langle x, Wx \rangle > 0 \) for any \( x \neq 0 \). It follows that \( W \) is a positive definite matrix. \( \square \)

The following theorem states that the sequence \( (w^{(k)}, p^{(k)}, z^{(k)}) \) generated by (10)-(13) with \( \theta = 1 \) will converge to the saddle point of \( J(w, p, z) \) under some assumption.

**Theorem 1.** Let \( \lambda = \| \nabla \text{div} \|_2^2 \) and \( (w^{(k)}, p^{(k)}, z^{(k)}) \) be the sequence generated by (10)-(13) with \( \theta = 1 \). Assume that \( J(w, p, z) \) in (9) has a saddle point \( (w^*, p^*, z^*) \). When the step sizes \( s, t \) satisfy the conditions \( s(\lambda + \beta) < 1 \) and \( t < \min(1, 1/\beta) \), respectively, there exists a saddle point \( (w^*, p^*, z^*) \) such that

\[
w^{(k)} \to w^*, p^{(k)} \to p^* \quad \text{and} \quad z^{(k)} \to z^*.
\]

**Proof.** A solution of \( J \) is a saddle point if and only if it satisfies the optimal condition

\[
\langle w - w^*, \nabla \text{div} p^* + \beta z^* \rangle_Y = 0 \quad (22a)
\]

\[
\langle p - p^*, \nabla f - \text{div} w^* \rangle_Y \geq 0 \quad (22b)
\]

\[
\langle z - z^*, -\beta w^* \rangle_Y \geq 0. \quad (22c)
\]
On the other hand, $w^{(k)}, p^{(k)}, z^{(k)}$ are the solution of the minimization problems in (10)--(13) if and only if they satisfy
\begin{align}
\langle w - w^{(k+1)}, \frac{1}{s}(w^{(k+1)} - w^{(k)}) + \nabla \text{div} p^{(k)} + \beta z^{(k)} \rangle_Y &= 0 \\
\langle p - p^{(k+1)}, \frac{1}{l}(p^{(k+1)} - p^{(k)}) + \nabla (f - \text{div} w^{(k)}) \rangle_Y &= 0 \\
\langle z - z^{(k+1)}, \beta \bar{W} - \beta I + z^{(k+1)} - z^{(k)} \rangle_Y &\geq 0
\end{align}
for any $w, p, z \in Y$.

For simplicity, we define
\[ x = \begin{pmatrix} w \\ p \\ z \end{pmatrix}, \quad \phi(x) = \begin{pmatrix} \nabla \text{div} p + \beta z \\ \nabla (f - \text{div} w) \\ -\beta w \end{pmatrix}. \]

Then, the optimal condition in (22) and (23) can be rewritten as
\[ \langle x - x^{*}, \phi(x^{*}) \rangle \geq 0 \] (24)
and
\[ \langle x - x^{(k+1)}, \phi(x^{(k+1)}) + W(x^{(k+1)} - x^{(k)}) \rangle \geq 0. \] (25)
respectively. Here, $W$ is the matrix defined in Lemma 1, which states that $W$ is a positive definition matrix under the assumption of the step size.

Now, we set $x$ to $x^{*}$ in (25) and $x$ to $x^{k+1}$ in (24), and then, sum the inequalities (24) and (25), we obtain
\[ \langle x^{*} - x^{(k+1)}, \phi(x^{(k+1)}) - \phi(x^{*}) \rangle + \langle x^{*} - x^{(k+1)}, W(x^{(k+1)} - x^{(k)}) \rangle \geq 0. \] (26)

Notice that
\[ x - x^{*} = \begin{pmatrix} w - w^{*} \\ p - p^{*} \\ z - z^{*} \end{pmatrix} \quad \text{and} \quad \phi(x) - \phi(x^{*}) = \begin{pmatrix} \nabla \text{div} (p - p^{*}) + \beta (z - z^{*}) \\ -\nabla \text{div} (w - w^{*}) \\ -\beta (w - w^{*}) \end{pmatrix}. \]

We have
\[ \langle x - x^{*}, \phi(x) - \phi(x^{*}) \rangle = 0. \] (27)

On the other hand, similarly to triangle identity of Euclidean geometry, we have
\[ \|x^{*} - x^{(k)}\|_{W}^{2} = \|x^{*} - x^{(k+1)}\|_{W}^{2} + \|x^{(k+1)} - x^{(k)}\|_{W}^{2} + \langle x^{*} - x^{(k+1)}, W(x^{(k+1)} - x^{(k)}) \rangle. \] (28)

Setting $x = x^{(k+1)}$ in (27) and then summing (26), (27), and (28), we have
\[ \|x^{*} - x^{(k)}\|_{W}^{2} \geq \|x^{*} - x^{(k+1)}\|_{W}^{2} + \|x^{(k+1)} - x^{(k)}\|_{W}^{2}. \] (29)

The inequality (29) implies that the sequence $\{\|x^{*} - x^{(k)}\|_{W}^{2}\}$ is both bounded and nonincreasing. It is known that a bounded nonincreasing sequence is convergent; therefore, the sequence $\{\|x^{*} - x^{(k)}\|_{W}^{2}\}$ is convergent. Hence, the sequence $\{x^{(k)}\}$ contains a subsequence $x^{(k)}_{j}$ converging to some limit point $x^{*}$. Notice that the inequality (29) also implies that
\[ \lim_{k \to \infty} (x^{(k+1)} - x^{(k)}) = 0. \]

In particular, $x^{(k+1)}_{j}$ also converges to $x^{*}$. Set $x^{(k)}_{j}$ in (23) and let $j \to \infty$, we obtain
\[ \langle x - x^{*}, \phi(x^{*}) \rangle \geq 0. \]
It implies that $x^*$ satisfies the optimal condition and, hence, is a saddle point of $\mathcal{J}$. Set $x^* = x^{\dagger}$ and assume that $\alpha$ is the minimum eigenvalue of the positive definition matrix $W$, we have $\alpha > 0$ and

$$\alpha \|x^* - x^{(k)}\|_2^2 \leq \|x^* - x^{(k)}\|_W^2.$$ 

Because the sequence $\{x^{(k)}\}$ contains a subsequence converging to $x^*$ and $\|x^* - x^{(k)}\|_W^2$ is nonincreasing, the whole sequence $\{x^{(k)}\}$ converges to $x^*$.

4 | NUMERICAL RESULTS

In this section, we illustrate the performance of our proposed algorithm for the image decomposition problem. Though several efficient numerical algorithms have been developed to solve the variants of the Meyer $G$-norm–based minimization problem, few works have been developed to solve the exact $G$-norm–based minimization problem due to the definition of the Meyer $G$-norm. More recently, the alternating direction method of multipliers (ADMM) and the nonlinear projection (NLP) method were applied to solve the exact $G$-norm–based minimization problem.\(^{25,29}\) We remark that the algorithms in the works of Aujol et al.\(^{25}\) and Ng et al.\(^{29}\) cannot be directly applied to solve the original Meyer image decomposition model. In fact, the ADMM was applied to solve the VO model in (5) with $p = \infty$,\(^{28}\) whereas Chambolle’s projection method was developed to solve the AABC model in (6). We will compare the proposed method with the ADMM and Chambolle’s projection method proposed in the works of Aujol et al.\(^{25}\) and Ng et al.\(^{29}\) The stopping criterion of all the methods is that the relative difference between the successive iterate of the restored image should be less than $5 \times 10^{-5}$ or the number of iterations reaches 1,000.

In order to measure the decomposition results, we generate two synthetic images. We generate the original cartoon image $u$ and the original texture image $v$ by the image decomposition algorithm such that $u$ and $v$ are consistent with Meyer’s model, where the cartoon component is well structured and the texture is modeled by oscillatory functions taking both positive and negative values, and of zero mean.\(^{15}\) Then, the synthetic image is superposed by a cartoon image and a texture image (see Figure 1). The quality of the decomposed images is assessed using the signal-to-noise ratio (SNR) and

![Figure 1](image-url)
Figure 2. The parameters are as follows: \( \alpha = 0.19, \beta = 0.12 \) for the alternating direction method of multipliers (ADMM) in the work of Ng et al., \( \alpha = 0.4, \mu = 31.1 \) for the nonlinear projection (NLP) method in the work of Aujol et al., \( \alpha = 0.4, \mu = 31.1 \) for the proposed method. The range of the pixel value of the images is normalized to \([0, 255]\). (a) \( \hat{u} \) (Ng et al.). (b) \( \hat{u} \) (Aujol et al.). (c) \( \hat{u} \) (proposed). (d) \( \hat{v} \) (Ng et al.). (e) \( \hat{v} \) (Aujol et al.). (f) \( \hat{v} \) (proposed).

Figure 3. The parameters are as follows: \( \alpha = 0.32, \beta = 0.20 \) for the alternating direction method of multipliers (ADMM) in the work of Ng et al., \( \alpha = 0.5, \mu = 21.38 \) for the nonlinear projection (NLP) method in the work of Aujol et al., and \( \beta = 1 \) for the proposed method. The range of the pixel value of the images is normalized to \([0, 255]\). (a) \( \hat{u} \) (Ng et al.). (b) \( \hat{u} \) (Aujol et al.). (c) \( \hat{u} \) (proposed). (d) \( \hat{v} \) (Ng et al.). (e) \( \hat{v} \) (Aujol et al.). (f) \( \hat{v} \) (proposed).
We also test other values of parameter \( \alpha \) in (5) and (6) tends to infinity, these two models coincide with the original Meyer model. In theory, \( \alpha \) should be chosen as large as possible. However, we find that, when we choose a large \( \alpha \), the SNRs for both decomposed cartoon image and the texture image become worse. Therefore, we apply a trial-and-error method to choose the best value of \( \alpha \) from the tested values such that the sum of the SNRs of the decomposed cartoon and texture images is the largest. Once \( \alpha \) is determined, we apply the similar rule to choose the parameter \( \beta \) in (5). The parameter \( \beta \) in the exact Meyer model (3) is fixed to 1. The parameter \( \mu \) in (6) is set the \( G \)-norm of the original texture image. The decomposed results are shown in Figures 2 and 3 with their largest SNRs in different image decomposition methods. We list the SNRs for the decomposed cartoon and texture images obtained by different methods in Table 1. The results show that the SNR of the proposed method for both the cartoon image and the texture image are better than those of the methods in the works of Aujol et al.\(^25\) and Ng et al.\(^29\)

Assume that \( f \) is the target image, and \( \hat{u} \) and \( \hat{v} \) are respectively the cartoon part and texture part, the errors for the decomposed images obtained by different methods are shown in Table 2. The residual errors \( \| f - \hat{u} - \hat{v} \|_2 \) for the method\(^29\) is small; we can observe that \( \| u - \hat{u} \|_2 \approx \| v - \hat{v} \|_2 \) for the method\(^29\) in Table 2. Because there is \( f = u + v = \hat{u} + \hat{v} \) for the proposed method, it is obviously \( \| u - \hat{u} \|_2 = \| v - \hat{v} \|_2 \). Although the method in the work of Aujol et al.\(^25\) is more efficient than that in the work of Ng et al.\(^29\) with respect to SNR measurement, it is poor in residual errors \( \| f - \hat{u} - \hat{v} \|_2 \) (see Figure 4.4). A possible explanation is that the VO model (5) is a relaxed version of the AABC model (6), whereas the methods in the works of Ng et al.\(^29\) and Aujol et al.\(^25\) are the numerical algorithm for the VO model and the AABC model, respectively.

Next, we apply the proposed method to two 512 × 512 images: “Barbara” image and “Dollar” image, as shown in Figure 5. We show the results of the proposed algorithm in Figures 6 and 7. The regularization parameter in (3) is set to \( \beta = 0.1 \) in the experiments. For the VO model in (5), we set \( \alpha = 0.1 \) and \( \beta = 0.01 \). For the AABC model in (6), we set \( \alpha = 10 \) and \( \mu = \| \hat{w}_\beta \|_\infty \), where \( \hat{w}_\beta \) is the solution obtained by the proposed method with regularization parameter \( \beta \). We also test other values of parameter \( \mu \); similar decomposed results are obtained. According to decomposed images in Figures 6 and 7, the proposed method provide better visual cartoon and texture components than the other two methods. In particular, we present the zoomed versions of the cartoon and texture components of “Barbara” image in Figure 6(g)–(l) and “Dollar” image in Figure 7(g)–(l). In the “Tablecloth” of “Barbara” image, the texture component generated by the

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Signal-to-noise ratios for the decomposed cartoon and texture images obtained by the alternating direction method of multipliers (ADMM) in the work of Ng et al.,(^29) the nonlinear projection (NLP) method in the work of Aujol et al.,(^25) and the proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>Cartoon ( \hat{u} )</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------</td>
</tr>
<tr>
<td>Tom &amp; Jerry</td>
<td>ADMM (( \alpha = 0.32, \beta = 0.24 ))</td>
</tr>
<tr>
<td></td>
<td>NLP(( \alpha = 0.25, \mu = 28.5 ))</td>
</tr>
<tr>
<td></td>
<td>Proposed (( \beta = 1 ))</td>
</tr>
<tr>
<td>Pig</td>
<td>ADMM (( \alpha = 0.32, \beta = 0.20 ))</td>
</tr>
<tr>
<td></td>
<td>NLP (( \alpha = 0.25, \mu = 21.4 ))</td>
</tr>
<tr>
<td></td>
<td>Proposed (( \beta = 1 ))</td>
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<tr>
<th>Table 2</th>
<th>Relative errors for the decomposed images obtained by the alternating direction method of multipliers (ADMM) in the work of Ng et al.,(^29) the nonlinear projection (NLP) method in the work of Aujol et al.,(^25) and the proposed method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method</td>
<td>ADMM</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>Tom &amp; Jerry</td>
<td>( | u - \hat{u} |_2 / | f |_2 )</td>
</tr>
<tr>
<td></td>
<td>( | v - \hat{v} |_2 / | f |_2 )</td>
</tr>
<tr>
<td></td>
<td>( | f - \hat{u} - \hat{v} |_2 / | f |_2 )</td>
</tr>
<tr>
<td>Pig</td>
<td>( | u - \hat{u} |_2 / | f |_2 )</td>
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<tr>
<td></td>
<td>( | f - \hat{u} - \hat{v} |_2 / | f |_2 )</td>
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</table>
FIGURE 4  The residuals of the decomposed images in Figures 4.1 and 4.2 by the alternating direction method of multipliers (ADMM) in the work of Ng et al.\textsuperscript{29} and the nonlinear projection (NLP) method in the work of Aujol et al.\textsuperscript{25} (a) $f - \hat{u} - \hat{v}$ (Ng et al.\textsuperscript{29}). (b) $f - \hat{u} - \hat{v}$ (Aujol et al.\textsuperscript{25}). (c) $f - \hat{u} - \hat{v}$ (Ng et al.\textsuperscript{29}). (d) $f - \hat{u} - \hat{v}$ (Aujol et al.\textsuperscript{25})

FIGURE 5  The original real images. (a) 512 $\times$ 512 Barbara image. (b) 512 $\times$ 512 Dollar note image

The proposed method is visually decomposed better than those by the other two testing methods. In the “One” symbol of “Dollar” image, the cartoon component generated by the proposed method is visually decomposed better than those by the other two testing methods.
FIGURE 6  The decomposition results for the Dollar image by the alternating direction method of multipliers (ADMM) in the work of Ng et al., the nonlinear projection (NLP) method in the work of Aujol et al., and the proposed method. The range of the pixel value of the images is normalized to $[0, 255]$. (a) $\hat{u}$ (Ng et al.). (b) $\hat{u}$ (Aujol et al.). (c) $\hat{u}$ (proposed). (d) $\hat{v}$ (Ng et al.). (e) $\hat{v}$ (Aujol et al.). (f) $\hat{v}$ (proposed). (g) $\hat{u}$ (Ng et al.). (h) $\hat{u}$ (Aujol et al.). (i) $\hat{u}$ (proposed). (j) $\hat{v}$ (Ng et al.). (k) $\hat{v}$ (Aujol et al.). (l) $\hat{v}$ (proposed).
FIGURE 7  The decomposition results for the Barbara image by the alternating direction method of multipliers (ADMM) in the work of Ng et al.,\textsuperscript{29} the nonlinear projection (NLP) method in the work of Aujol et al.,\textsuperscript{25} and the proposed method. The range of the pixel value of the images is normalized to \([0, 255]\).

(a) \(\hat{u}\) (Ng et al.\textsuperscript{29}). (b) \(\hat{u}\) (Aujol et al.\textsuperscript{25}). (c) \(\hat{u}\) (proposed). (d) \(\hat{v}\) (Ng et al.\textsuperscript{29}). (e) \(\hat{v}\) (Aujol et al.\textsuperscript{25}). (f) \(\hat{v}\) (proposed). (g) \(\hat{u}\) (Ng et al.\textsuperscript{29}). (h) \(\hat{u}\) (Aujol et al.\textsuperscript{25}). (i) \(\hat{u}\) (proposed). (j) \(\hat{v}\) (Ng et al.\textsuperscript{29}). (k) \(\hat{v}\) (Aujol et al.\textsuperscript{25}). (l) \(\hat{v}\) (proposed).
5 | CONCLUSION

In this paper, we have developed a numerical algorithm to solve the exact Meyer G-norm–based image decomposition problem. To handle the difficulty caused by the infinity norm, we represented it using a dual formulation and then applied a primal-dual algorithm to compute the solution. An advantage of the proposed method is that there exists a closed-form solution for each subproblem, and therefore, no inner iterations are needed in the proposed method. The convergence of the numerical algorithm is also studied and shown. Numerical results have shown that the exact Meyer model can give better image decomposition results than the other testing methods.

ACKNOWLEDGEMENTS

Y. Wen’s research is supported in part through NSFC Grant 11871210, the Construct Program of the Key Discipline in Hunan Province, and the Scientific Research Fund of Hunan Provincial Education Department Grant 17A128. M. Ng’s research is supported in part through the HKRGC GRF 1202715, 12306616, 12200317, 12300218, and HKBU RC-ICRS/16-17/03.

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How to cite this article: Wen Y-W, Sun H-W, Ng MK. A primal-dual method for the Meyer model of cartoon and texture decomposition. Numer Linear Algebra Appl. 2018;e2224. https://doi.org/10.1002/nla.2224