Block Investment and Partial Benefits of Corporate Control

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Despite familiar arguments for diversification, many investors choose to hold significant blocks of equity in the same firm. While control benefits may explain majority blocks, most blocks are much smaller than what is generally considered necessary for control. This paper develops a theory whereby such blocks can confer partial benefits of control to their holders; in particular, small block shareholders can join together and form controlling coalitions. The implications of such a cooperative game among block shareholders for the shareholder structure within and across firms are examined. This paper predicts large investors will "create their own space" by staking out large enough blocks to deter other block investors, there will be a threshold level above which large investors are not challenged, and that the shareholder structure across firms will exhibit a particular clientele effect among block shareholders. These predictions are consistent with a preliminary review of empirical evidence.

1. INTRODUCTION

Despite theoretical recommendations for diversification, many investors choose to hold significant blocks of equity in the same firm. Fortune 500 corporations in 1981 had on average 10.5 shareholders with blocks exceeding 1% of their equity, 4.7 with blocks greater than 2%, and 1.4 with blocks greater than 5%. Furthermore, these block shareholdings exhibit patterns across firms not explained by standard investment theory. The largest shareholders tend to "create their own space"; their presence seems to dissuade other large shareholders from investing in their firm. Similarly, the larger the leading shareholder in a firm, the fewer smaller block shareholders are present, and firms without a large leading shareholder tend to have a greater number of moderate sized shareholders. This paper attempts to explain this "shareholder structure" through a model in which block investors receive private benefits from partial control, by forming controlling coalitions which can divide these benefits.

Previous considerations of control contests have generally considered the battle between an incumbent and a potential raider, taking other shareholders as passive observers whose only actions are their tender or their vote. Additionally, these shareholders are typically assumed to be sufficiently dispersed so that each one individually has a negligible effect on the outcome of a control contest. Such a setting yields the Grossman and Hart (1980) free-rider problem, whereby a superior rival cannot take over a firm at less than its public value, because non-tendering shareholders would realize this value upon a successful takeover. Much recent literature on control contests can be viewed as an attempt to explain the raider's motivation for a takeover, and the process through which such a contest is played out, in light of this free-rider problem. Several different

1. These numbers are for the 456 firms covered in the 1981 CDE Stock Ownership Directory: Fortune 500. The 44 excluded firms had either recently merged, were privately held, or were subsidiaries or cooperatives.
manner in which raiders can profit despite free-rider problems have been proposed: Grossman and Hart (1980) consider the ability of a raider to dilute a firm for private gain; Shleifer and Vishny (1986) and Hirshleifer and Titman (1990) examine a raider’s ability to profit on the appreciation on a foothold held prior to a takeover attempt; and Grossman and Hart (1988), Harris and Raviv (1988a), Stulz (1988) and Dewatripont (1993), among others, all consider private benefits to control.

Shareholders in Harris and Raviv (1988b) and Holmstrom and Nalebuff (1992) do take into account the possibility that their tender decision may be pivotal, which in turn affects equilibrium bids of the control contestants. However, beyond this tender decision, shareholders once again play a passive role; the possibility that they can coordinate with control contestants is not considered. If instead such coordination between the raider and any one of the shareholders was possible (for example, if a shareholder could secretly sell a few shares to the raider, or could promise to tender in exchange for a payment), both could gain considerably if others continued to undertake equilibrium actions.

Thus, while this existing literature has gone far in analysing and understanding the market for corporate control, it only yields a rationale for block shareholding for direct control contestants, and fails to address both why the great majority of block shareholders hold blocks significantly smaller than majority blocks, and the role these investors play in control contests. Indeed, many firms lack the majority or dominant shareholder often presumed in the literature, while at the same time, taken together, block shareholders hold a sizeable fraction of equity. Motivated by these observations, together with the increasingly visible role large institutional investors and organizations such as CalPERS and the Council of Institutional Investors have played in corporate governance, this paper departs from the literature by considering an active role played by block shareholders as a central component in determining corporate control. Shareholders’ role as active participants in controlling coalitions in turn gives them an incentive to hold blocks.

While private benefits of control have received much attention recently, much of this literature is vague on the origins of these benefits. Among plausible sources are the ability of management (or directors) to dilute corporate funds for private benefit, synergies obtainable through mergers, favours conferred by a firm, access to inside information, perquisites of control, and utility derived directly from power or control. While most previous papers have modelled these control benefits as indivisible, all these sources are likely to yield benefits shared by a number of individuals.

If benefits of control are indeed divisible, it is natural to presume that the degree of control an investor derives from a block will depend on the strategic importance of this block in forming controlling coalitions. If one investor has a majority position, a moderate-sized block investor in this same firm is likely to obtain few if any control benefits; if

2. Of the 10.5 shareholders holding greater than 1% of the average Fortune 500 firm’s equity, only 0.54 of them hold greater than 10% of the firm. The median largest shareholder in these firms holds only about 9%, and the mean largest shareholder 15.4%, whereas the mean top 5 shareholders hold 28.8%. These figures are derived from the 1981 CDE Stock Ownership Directory: Fortune 500; the latter two means are also reported in Shleifer and Vishny (1986). Demsetz and Lehn (1985) find that the top 5 shareholders held on average 24.8% of a firms’ equity, and the top 20 held 37.7%, for a slightly different data set.

3. Jensen and Ruback (1983), Demsetz (1986), Grossman and Hart (1988), Harris and Raviv (1988a and b), Barclay and Holderness (1989), Bergström and Rydqvist (1990), Hart and Moore (1990), and Dewatripont (1993), among others, all model or discuss such benefits. See Barclay and Holderness for a good discussion on the plausibility and source of private benefits. Additionally, there exists empirical evidence suggesting that control is valuable. Lease, McConnell and Mikkelson (1983), Levy (1983) and Zingales (1994) find that for firms with dual classes of equity identical except for their voting privileges, the superior voting class sells at a premium, in respectively the U.S., Israel and Italy. Barclay and Holderness find that on average, large premia (greater than 20%) relative to post-trade market price are paid for large blocks of equity.
instead other shares are held by many disperse individuals, a moderate-sized block may confer a sizeable degree of control. Large investors would like to invest their money across firms in a manner that maximizes benefits from control, understanding that others are acting likewise. This paper derives equilibrium implications for the shareholder structure within and across firms attributable to such strategic considerations.

The model makes several strong predictions for the shareholder structure which are distinct from previous models of block shareholders. First, large investors "create their own space"; the presence of a large block in a firm deters other large blocks from locating in the same firm. Second, the model predicts a clientele effect in the shareholder structure. In equilibrium there are three different types of shareholder structures: firms with one very large shareholder and no smaller block shareholders, firms with one large shareholder and many smaller block shareholders, and firms consisting of numerous small block shareholders but no dominant shareholder. Additionally, the larger the leading block shareholder, the fewer smaller block shareholders will be present in the firm. And third, the model predicts that there will be a threshold size, above which a large block will not be challenged for control.

A preliminary empirical examination—presented in Section 4—indicates that these predictions do correspond with empirical tendencies among block shareholders. Additionally, there is considerable anecdotal evidence suggestive of block shareholders actively pursuing private benefits. During takeover contests, opposing sides actively recruit block shareholders. Pound (1988) finds that block shareholders generally side with management during proxy fights, and that management tries to influence the votes of money funds. Brickley, Lease, and Smith (1988) find that institutional and block shareholders vote more often than other shareholders, and institutions noted for business ties with firms in which they invest (banks, insurance companies and trusts) are more likely to vote with management than other institutions (mutual funds, foundations and public pension funds). And, perhaps most significantly, numerous recent examples of corporate activism by institutional investors can be interpreted as a manifestation of a coalition of block shareholders opposing (or supporting through passivity) management.

Section 2 presents a general model to analyse strategic investment decisions when control benefits are divisible. Section 3 analyses equilibria of this model. Section 4 presents several examples and preliminary empirical evidence, and Section 5 suggests extensions and concludes.

2. THE MODEL
We now consider a model of financial investment when there exist divisible control benefits. Consider an economy with \( J \) identical firms, each with a single class of equity. Total private benefits to control of each firm is 1. In order to capture the notion that the division of benefits within a firm should correspond to shareholders' strategic importance in forming winning coalitions, we employ the Shapley values of a normalized cooperative

4. Such activity has been highlighted recently in a number of well publicized proxy contests. For example, in the Lockheed proxy contest, over one dozen institutional investors agreed to support the challenger Harold Simmons in exchange for a larger voice in the new management. Incumbent management defeated Simmons only after granting institutional investors three seats on the board and an expanded role in management. Over the last several years, pressure from institutional investors has instigated changes in management at IBM, American Express, Sears, Westinghouse, and General Motors, among others.
majority voting game to represent this division. While Shapley values are employed for the analytical simplicity this division yields, qualitative results can be shown to hold for any specification for the division of control benefits that satisfies a set of general assumptions given in Appendix B and detailed in Zwiebel (1991).

In order to explore the interaction between different-sized investors, initially two sizes of risk-neutral block investors are modelled. \( N \) agents are endowed with wealth \( n \), and \( M \) with wealth \( m \); where wealth is expressed in firm-size units. We assume there is no borrowing of funds, and all wealth is invested. To distinguish between the two types of investors, we denote the investors of size \( n \) as "type 1 investors", and those of size \( m \) as "type 2 investors", and let \( s = n/m \). The \( N \) type 1 investors may be considered "very large investors" capable of "dominating" one firm, while the \( M \) type 2 investors are large enough to hold significant blocks and participate in coalitions, but not large enough to dominate a firm. We assume \( n > m \), \( N < J \), and \( M > J \) to reflect this interpretation and to correspond with empirical observation. Furthermore, we assume the wealth of all block investors combined is strictly smaller than the entire market.

All shares not held by block investors are held by liquidity traders, who are taken to be too small individually to acquire blocks and obtain any benefits of control. These traders play two roles in the model. First, they provide liquidity. Since they obtain no private benefits, they are willing to buy and sell at the firms' public valuations (which is identical across all firms). This allows us to consider large investors' strategies solely as an allocation problem. Large shareholders' objective is hence solely to allocate their wealth across firms to maximize their control benefits.

5. The cooperative game solution concept of Shapley values is the unique outcome that satisfies three simple axioms (symmetry, linearity, and efficiency of strategically relevant agents). Explicitly, the Shapley value of agent \( i \) is given by,

\[
\phi_i = \sum_{T \subseteq N, i \in T} \frac{(t-1)!(n-t)!}{n!} \left[ v(T) - v(T - \{i\}) \right],
\]

where \( v \) represents the characteristic function for the game, \( N \) the set of all \( n \) players, and \( t \) the number of players in \( T \). This expression is simply the expectation, taken over an equal weighting of all possible orderings of players, of the marginal contribution of agent \( i \) to the set of preceding players in the ordering. For more detail, see Owen (1982).

Loosely speaking, this outcome can be interpreted as the mutual agreement by all agents on how to split net surplus, allowing for side payments, taking into account one another's strategic importance. Alternatively, and more rigorously, non-cooperative game foundations for Shapley values have been given by Gul (1989) and Hart and Mas-Colell (1992). In particular, both these papers provide simple non-cooperative dynamic bargaining games such that agents obtain their Shapley values in expectation. In Stole and Zwiebel (1994), Shapley values are obtained for a particular class of games as the equilibrium of a non-cooperative bargaining game without taking expectations.

6. Proposition 4 extends results to allow for an arbitrary number of different-sized investors. Allowing for risk aversion would imply that in equilibrium, instead of holding single blocks, shareholders would hold several blocks, trading off standard diversification benefits with the control related concentration benefits of the model. This will not alter any of the main implications for the shareholder structure.

Allowing investors to borrow at an increasing cost will lead them to borrow to the point at which the marginal cost from further borrowing equals the marginal benefit of greater concentration. Then wealth levels \( n \) and \( m \) can be interpreted as wealth after such an optimal borrowing decision, and once again results concerning the shareholder structure are valid. Similarly, if instead investors can borrow at a fixed cost on margin using shares as collateral, wealth levels can be interpreted as maximal investment when the margin requirement is binding. Of course, one straightforward justification of an increasing cost of borrowing (in order to invest in the market) is risk aversion. The possibility of the firm adjusting leverage in order to affect the amount of wealth needed to obtain a given sized block is discussed in the conclusion.

7. Implicitly, we assume here that liquidity traders hold some shares of all firms. Provided that liquidity traders collectively comprise a "large enough" share of the market, this will be true in equilibrium. An interesting extension to this model would be to consider the effect of dual equity classes with differential voting rights, which would serve to separate liquidity traders and block shareholders.
Secondly, some liquidity traders vote randomly, thereby creating noise in the outcome of close control contests and smoothing the value of control to large shareholders. Let the liquidity vote be given by $2\beta$, with $\beta + \bar{e}$ votes for a given coalition and $\beta - \bar{e}$ voting against. We assume the random variable $\bar{e}$ is distributed uniformly on support $[-m/2, m/2]$. Without these noise votes, the Shapley value of a participant in a normalized simple voting game can be given by the probability over equally-weighted random orderings that she will be the swing voter; with noise, expected benefits are instead given by this probability taken jointly over equally-weighted random orderings and noise.

The following stylized story is suggestive of our setting. Shareholders first choose where to invest, and then, shareholders of each firm gather at an annual meeting. Small costs prevent all but block shareholders from attending. At the meeting, shareholders vote on which slate of directors to support and what policies these directors will undertake, in the process implicitly specifying how benefits of control are divided. Prior to the vote, shareholders present attempts to form winning coalitions. Non-attendees may vote by proxy, but cannot participate in this coalition formation.

Finally, we consider an order of play and an equilibrium concept that captures the notion that small blocks are more liquid than large blocks, and therefore, type 2 investors (who can only purchase smaller blocks) can react to type 1 shareholders' investments. We will in fact envision investment in small blocks to be completely liquid, and therefore effectively allow type 2 investors to react to one another as well. In particular, it is assumed that type 1 investors (who are large enough to purchase large blocks, and will do so in equilibrium) allocate their wealth before type 2 investors. Attention is then restricted to pure-strategy subgame-perfect equilibria (PSSPE), as equilibria in pure strategies for type 2 shareholders will be stable in the sense that none of them would desire to re-invest. Our timing assumptions and equilibrium concept can therefore be interpreted as requiring a steady state to the investment game when there are small transaction costs.
on large block transactions which are not incurred on small block transactions. We will also consider refinements of PSSPE based on this notion of liquidity.

It will be useful to first consider, in the following three lemmas, control benefits for several cases which play an important role in the analysis below. The first simply states that re-scaling the size of all voters (including net liquidity voters) does not change the outcome within a firm; its proof follows immediately from the definition of Shapley values and the manner in which noise is added. The other two characterize how benefits of control are divided in firms with one or two large shareholders and \( L \) smaller identical block shareholders.\(^{13}\) Let \( \phi(s_1, s_2, \ldots, s_L, z) \) denote the \( I \times 1 \) vector of expected control benefits when there are \( I \) individuals in a firm of sizes \( s_1, s_2, \ldots, s_L \), and noise is distributed as \( \varepsilon \sim U[-z, z] \).

**Lemma 1.** \( \phi \) is homogeneous of degree 0.

**Lemma 2.** Suppose there is one individual of size \( x \) and \( L \) of size \( y \) in a firm, noise votes are distributed as \( \varepsilon \sim U[-y/2, y/2] \), and let \( s \equiv x/y \). Then letting \( \phi_1 \) and \( \phi_2 \) denote the expected benefits to control of the investor of size \( x \) and the investors of size \( y \) respectively, it follows that,

\[
\phi_1 = \min \left\{ \frac{s}{L+1}, 1 \right\}
\]

(1)

\[
\phi_2 = \max \left\{ \frac{L+1-s}{L(L+1)}, 0 \right\}.
\]

(2)

**Lemma 3.** Suppose there is one individual of size \( x_0 \), one of size \( x_1 \) and \( L \) of size \( y \) in a firm, and the noise from noise traders' votes is distributed as \( \varepsilon \sim U[-y/2, y/2] \). Define \( s_i \equiv x_i/y \), \( i = 0, 1 \). Then the expected benefits of control to investors of size \( x_0 \) and \( x_1 \) are given by,

\[
\phi_i = \begin{cases} 
\frac{s_i(L-s_i+2) + (\sigma_{ij}^2 - \sigma_{ii}^2)}{(L+2)(L+1)} & \text{if } s_0 + s_1 \leq L + 1 \\
\frac{(L-s_j+s_i+2)^2 - 1 + 4\sigma_{ij}^2}{4(L+2)(L+1)} & \text{if } s_0 + s_1 > L + 1 \\
& \text{and } s_0 < L + s_1 + 1 \\
1 & \text{if } s_j > L + s_i + 1 \\
0 & \text{if } s_j > L + s_i + 1 
\end{cases}
\]

(3)

where \( i,j = 0,1, i \neq j \), and \( \sigma_{ij}^2 \) and \( \sigma_{ii}^2 \), defined in equation (A12) in the proof, are continuous periodic functions of \( L - s_j + s_i \) and \( L - s_i - s_j \) with period 2, that takes values between 0 and \( 1/4 \).

The particularly simple form that benefits take in Lemma 2 follows from our distributional assumption for noise votes, which serves to smooth benefits in a simple and intuitive manner. Other continuous distributions of noise would alter this simple characterization.

\(^{13}\) All proofs are in the Appendix.
without qualitatively affecting our results. The distributional assumption employed here in turn gives rise to the $\sigma_{Lj}^2$ and $\sigma_{Sj}^2$ terms in Lemma 3. These artifacts of noise are small relative to other terms, and inconsequential to what follows.\(^1\)\(^4\) Alternatively, the divisions of control benefits in Lemmas 2 and 3 follow when noise is large relative to type 2 investors (and still uniform), and the size of type 1 investors is not exceedingly large.\(^1\)\(^5\) Furthermore, as $y \to 0$ and $L \to \infty$ such that $yL$—the total size of type 2 shareholders—remains constant, benefits to type 1 shareholders under Lemmas 2 and 3 approach those when there is no noise (which is in turn equivalent to oceanic Shapley values).

Lemma 2 yields a division of benefits easy to interpret. When $s = 1$, individual 1 is the same size as the other $L$ shareholders and thus receives identical control, given by $\phi_1 = 1/(L + 1)$. At $s = L + 1$, individual 1 always wins any vote (even with the worse possible draw of noise), hence $\phi_1 = 1$. Between these values, $\phi_1$ is linear in $s$. Thus, marginal benefits to concentration for the large shareholder are constant over this range.\(^1\)\(^6\) As $y \to 0$ and $L \to \infty$ so that $yL$ remains constant, benefits to the large shareholder $\phi_1$ grow, and total benefits to the smaller block shareholders $L\phi_2$ fall. Intuitively, investor 1 prefers other blocks to be broken up, as this increases his possibility of forming winning coalitions.

Lemma 3 yields a similar interpretation. Ignoring the small $\sigma^2$ terms, $\phi_1$ is once again linear in $s_1$ when $s_0 + s_1 \leq L + 1$. When instead $s_0 + s_1 > L + 1$, and therefore the two large shareholders can form a winning coalition by themselves, $\phi_1$ is convex in $s_1$. Additionally, if $s_0 = 0$ (shareholder 0 is not present), then $\phi_1 = s_1/L + 1$, and if $s_0 = 1$ (shareholder 0 is the same size as the other $L$ shareholders), then $\phi_1 = s_1/L + 2$, which are consistent with Lemma 2.

3. MARKET EQUILIBRIUM

We now consider the market equilibrium with $J$ firms, $N$ type 1 shareholders of size $n$ and $M$ type 2 shareholders of size $m$. As discussed above, attention is restricted to pure-strategy subgame-perfect equilibria. Proposition 1 states that PSSPEs exist, and have a simple appealing form, which depends on the relative size $s$ of type 1 vs. type 2 shareholders.

**Proposition 1.** For given values of $J$, $N$, and $M$, there exists values $1 \equiv s_{N+1} < s_N < s_{N-1} < \cdots < s_1 = \bar{s}$, such that for $s$ satisfying $s_k < s \leq s_k$, PSSPE are as follows:

Each of the type 1 agents invests all wealth in a different firm. Let $J(N) \subset J$ denote the set of firms with a type 1 investor. Each type 2 investor also concentrates all wealth in one firm. Denoting the number of type 2 investors in firm $j$ by $L(j),$ \[ L(j) = \begin{cases} M_1 & \forall j \in J \setminus J(N) \\ M_2 & \forall j \in J' \subseteq J(N) \\ 0 & \forall j \in J(N) \setminus J' \end{cases} \] (4)

where $J'$, the set of firms with type 1 individuals who are challenged, is any subset of $J(N)$.

\(^1\) Continuity of $\phi_1$ in Lemma 3 follows from noting that $s_0 + s_1 = L + 1$ implies $\sigma_{Lj}^2 = 0$, and $s_1 = L + s_j + 1$ implies $\sigma_{Sj}^2 = 0$.

\(^1\)\(^5\) In particular, provided that $s$ is not "too close" to $L + 1$ in Lemma 2 and $s_1$ is not "too close" to $L + s_j + 1$ in Lemma 3.

\(^1\)\(^6\) This should not be confused, however, with the incorrect statement that control benefits are proportional to a shareholder's fractional share of total block holdings. Indeed, the fraction of all block shares held by individual 1, given by $s/(L + s)$, is concave in $s$, and consequently $\phi_1$ is convex in this fraction over the range $[0, 1/2]$.
of cardinality $|J'| \leq k$ (each such $J'$ characterizes a different PSSPE), and $M_1$ and $M_2$ are constants satisfying the following conditions:

$$M_1(J-\mathbb{N}) + M_2J' = M$$  \hspace{1cm} (5)$$

$$\phi^t(M_1) = \phi^t(M_2) \quad \forall j \in J \setminus J(\mathbb{N}), \forall j' \in J'$$  \hspace{1cm} (6)$$

$$\frac{\partial \phi^t}{\partial L}(L(j)) \leq 0 \quad \forall j \in J \setminus J(\mathbb{N}), \forall j' \in J';$$  \hspace{1cm} (7)$$

where $\phi^t(L)$ denotes expected control benefits for a type 2 investor in firm $j$ when there are $L$ type 2 investors in that firm.  \hspace{1cm} (8)

While this proposition is long, what it states is intuitive. First, type 1 shareholders invest all their wealth in one firm. (Recall that risk-aversion motives for diversification have not been built into the model.) Intuitively, this follows from the weak convexity of payoffs in size for the large shareholder (Lemmas 2 and 3), reinforced by the fact that a larger stake is challenged by fewer type 2 shareholders. Second, type 2 shareholders react by also investing all wealth in one firm. And third, and most significantly, type 2 shareholders distribute themselves across all firms without type 1 shareholders and challenge a subset $J' \subseteq J(\mathbb{N})$ of firms with type 1 shareholders, in a manner that equates the benefits they receive from all firms in which they invest. In equilibrium, there are three types of firms; those without any dominant shareholders but rather many shareholders holding smaller blocks, those with one large shareholder who is uncontested, and those with one large shareholder who is contested by smaller block shareholders.

Type 2 shareholders contesting any number of type 1 shareholders $|J'|$ less than or equal to $k(s)$ yields an equilibrium; where $k(s)$ is a decreasing function of $s$ defined by the critical values $s_{N+1}, s_N, \ldots, s_1$ of Proposition 1. Contesting greater than $k(s)$ of these firms is not an equilibrium. Successfully challenging more than $k(s)$ type 1 shareholders would require so many type 2 shareholders that those left in firms without a type 1 shareholder would do better than the challengers, thereby inducing challengers to switch out. Since the number of type 2 shareholders needed to challenge a type 1 shareholder increases with $s$, the maximal number of firms that can be challenged in equilibrium falls with $s$.

If $s$ exceeds $\bar{s}$, it follows that $k = 0$, and then only $J' = \emptyset$ is an equilibrium (no type 1 investors are contested). Since type 1 investors are not challenged beyond this size, they will cease investing all their wealth in one firm. Instead, they will invest $\bar{m}$ in the first firm in which they create a stake, and invest remaining wealth in another firm (up to $\bar{m}$, and if they still have more wealth, invest elsewhere, etc.).

17. When the meaning is clear, we let $J'$ also denote the cardinality of this set.

18. This proposition does not characterize all PSSPE for two reasons. The first is an integer problem; equilibria exist with $M_1$ investors in some firm $j \notin J' \setminus J(\mathbb{N})$ and $M_1 + 1$ in another firm $j' \notin J' \setminus J(\mathbb{N})$, where individuals in the former do not switch to the latter because then the latter would have $M_1 + 1$ individuals. More accurately, this proposition and others throughout the paper characterize equilibria to within such integer roundings.

A second reason that this proposition does not characterize all equilibria is because "pathological" play by type 1 shareholders can be induced by choosing off-the-equilibrium play by type 2 shareholders appropriately. In particular, as the proposition states, there are equilibria in which any number from 0 to $k$ type 1 investors are challenged by type 2 investors. The fewer type 1 investors are challenged, the better off they are. Thus, equilibria exist of the following nature: type 1 investors diversify their holdings, because if they do, type 2 investors play an equilibrium good for them, where no large investors are challenged; whereas if they do not, type 2 individuals play an equilibrium bad for them, where they challenge $k$ type 1 investors. However, all such equilibria fail the cooperative refinements of Proposition 2, under which type 2 investors always challenge the maximum number of type 1 shareholders $k$ that they can.
Benefits to a type 2 shareholder as a function of the number of type 2 shareholders in a firm are depicted in Figure 1. The higher curve represents benefits in a firm without a type 1 shareholder, the lower curve benefits with a type 1 shareholder. Note the single-peaked shape of the latter. Intuitively, as the number of type 2 shareholders challenging a type 1 shareholder increases, they are better able to contest the type 1 shareholder, but must split these increasing joint benefits among more of themselves. As \( M_2 \) exceeds \( s - 1 \), initially the marginal benefits from challenging exceed the average benefits per challenger and hence \( \phi_2 \) (which equals the average benefits) increases with \( M_2 \). Eventually, however, the reverse is true and \( \phi_2 \) decreases with \( M_2 \). This single-peaked shape of \( \phi_2 \) in firms with a type 1 shareholder consequently induces two solutions to equations (5) and (6), given by equation (A17) in the Appendix and depicted by the pair of dashed lines connecting the \( M_1 \) and \( M_2 \) curves in Figure 1. At both of the pairs of values for \( M_1 \) and \( M_2 \) connected by the lines, benefits are equated across firms and all type 2 shareholders have allocated wealth in some firm. Such a pair will represent an equilibrium provided that benefits to type 2 shareholders within all challenged firms decrease with more type 2 shareholders; that is, provided that equation (7) is satisfied as well. It is clear from the diagram that this can only be possible for the larger root of (A17).

Of all the equilibria in Proposition 1, the one where type 2 shareholders challenge the maximal number of type 1 shareholders stands out as particularly appealing. Specifically, this is the equilibrium most desirable for type 2 shareholders and the outcome that would be obtained if they coordinated their actions. Intuitively, many PSSPEs may exist because it never pays for one type 2 shareholder to unilaterally challenge a larger type 1 shareholder, but a group of type 2 shareholders may benefit by doing so. Even maintaining a non-cooperative perspective when investment decisions are made, this particular outcome is natural under our notion of costless small block reinvestment. In a PSSPE that is not
optimal for type 2 shareholders, a type 2 shareholder could costlessly challenge a type 1 shareholder and test whether others will follow such a deviation. And other type 2 shareholders could costlessly join in, until enough have done so to make the deviations profitable for all of them.

Confirming this intuition is the following Proposition, which indicates that this equilibrium uniquely survives several simple cooperative refinements. In particular, it is uniquely chosen by the relatively weak cooperative refinement of Perfect Coalition-Proof Nash Equilibrium (PCPNE) (Bernheim, Peleg, and Whinston (1987)) and yet, still survives a much stronger refinement we call Class Strong Equilibrium (CSE)—a modification to Strong Equilibrium (Aumann (1959)) appropriately chosen to capture the dynamic structure of the game.  

**Proposition 2.** There exists a unique PSSPE which survives the refinements of PCPNE and CSE. This equilibrium is the PSSPE characterized in Proposition 1, with the maximal number of firms $k$ contested both on and off the equilibrium path. That is, for $s_k + 1 < s \leq s_k$, $|J'| = k$, and for $s > \bar{s} = s_1$, $J' = \emptyset$.

Intuition for this result follows from Figures 2 and 3, and Lemmas 4 and 5 (which are used in proving the proposition). Figures 2 and 3 depict the distributions of type 2 shareholders and their corresponding benefits in PSSPEs with different numbers of firms challenged when $J = 500$, $N = 200$, $M = 10,000$ and $s = 5$ (Figure 2) and $s = 8$ (Figure 3). Points on the curves represent benefits per type 2 shareholder in the equilibria where the labelled number of firms are challenged. When $s = 5$, there exists an equilibrium where all firms are challenged. When instead $s = 8$, in the equilibrium where the maximum number of firms are challenged, the number of type 2 challengers per firm maximizes, to within integer roundings on $k(s)$, possible benefits against a type 1 shareholder of this size (obtaining the peak of $\phi_L$, $j \in J(N)$). In general, one of these two outcomes must always occur, as the following lemma states.

**Lemma 4.** For any given $s$, in the equilibrium where the maximal number of firms $k(s)$ are challenged; either $k(s) = N$; or to within integer roundings on $k$, when $k(s)$ firms are challenged, $M_2 = \text{argmax}_L \left( \phi_L^*(L), j \in J(N) \right)$.

More generally, the following lemma indicates that type 2 shareholders do better, and type 1 shareholders worse, in equilibria with more type 1 shareholders challenged.

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19. In particular, we define a Class Strong Equilibrium as an equilibrium profile that is immune to all Pareto-improving deviations by any coalition that are in turn immune to Pareto-improving counter-deviations in a future period. In our setting, this rules out equilibrium breaking deviations between type 1 and type 2 shareholders that would in turn lead to reactions by the more liquid type 2 shareholders that make the type 1 shareholders worse off (and the type 2 shareholders better off). (Hence the name Class Strong Equilibrium, as only deviations within the same types of investors can break the equilibrium.) Such a restriction seems a natural extension of perfection to Strong Equilibrium in dynamic games. Note that this does not rule out deviations that are not immune to counter-deviations in the same period, and as such, maintains the essential difference between the strong equilibrium concept and the less restrictive coalition-proof refinement. Restrictions of deviations fragile to future counter-deviations are naturally built into the PCPNE concept where deviations must be immune to all counter-deviations (that are in turn immune to counter-deviations, etc...), but are lacking in the very stringent concept of Perfect Strong Equilibrium (Rubinstein (1980)), which requires that the equilibrium be strong in every subgame. It is on this point that Bernheim, Peleg, and Whinston find Perfect Strong Equilibrium to be "too strong". It is worth noting that while Strong Equilibria do not exist for essential constant sum games, a CSE can exist (and presently does exist), because the game induced on type 2 agents by holding the play of type 1 agents fixed is not constant sum.
FIGURE 2
$s=5$; All firms can be challenged in Equilibrium

FIGURE 3
$s=8$; Not all firms can be challenged in Equilibrium
Hence, the PSSPE selected by Proposition 2 (with the maximal number of firms $k(s)$ challenged), is the optimal PSSPE for type 2 shareholders.

**Lemma 5.** Comparing PSSPEs, as the number of firms challenged in equilibrium increases, $M_1$ and $M_2$ decrease and $\phi_2$ increases.

While the characterization of equilibria distributions of type 2 shareholders given by (A17) in the Appendix appears quite unwieldy, a number of simple comparative statics results follow immediately from this expression.

**Proposition 3.**

1. As $s \to 1$; $M_2 \to \frac{(M - (J - N))}{J - N + J'}, \ M_1 \to \frac{J' + M}{J - N + J'}$, and $(M_1 - M_2) \to 1$.
2. $J = N \Rightarrow M_2 = M/J'$.
3. $M_1(M, N, J, J'; s)$ and $M_2(M, N, J, J'; s)$ are homogeneous of degree 0 in $(M, N, J, J')$.
4. In the CSE of Proposition 2, $\forall s$,
   \[
   \frac{\partial M_1}{\partial s} > 0, \quad \frac{\partial (J'M_2)}{\partial s} < 0, \quad \frac{\partial \phi_1}{\partial s} > 0, \quad \frac{\partial \phi_2}{\partial s} < 0.
   \]
5. In the CSE of Proposition 2, besides at critical values $s_N, s_{N-1}, \ldots, s_1$ where $J'$ falls and therefore $M_2$ jumps up,
   \[
   \frac{\partial M_2}{\partial s} < 0.
   \]
6. Comparing different PSSPE of Proposition 1, $\phi_2$ increases with $J'$, and both $M_1$ and $M_2$ decrease with $J'$.

Proposition 3.1 states that as type 1 shareholders' size approaches that of type 2 shareholders, type 2 shareholders distribute themselves symmetrically across all firms they challenge (which is all firms for the CSE as $s < s_N$ and therefore $J' = N$), save for taking into account that there is already one shareholder present in firms of $J(N)$. When $s = 1$, this results in all firms having the same number of identical-sized shareholders. Proposition 3.2 indicates that type 2 shareholders also distribute themselves evenly across all firms when there is one large type 1 shareholder in each firm, while Proposition 3.3 points out that the equilibrium distributions of type 2 shareholders are unaffected by changes in scale. Proposition 3.4 and 3.5 state that in the CSE, as the size of type 1 shareholders grows relative to the size of type 2 shareholders: type 1 shareholders do better, type 2 shareholders do worse, the number of type 2 shareholders in any firm without a type 1 shareholder rises, the total number of type 2 shareholders challenging type 1 shareholders falls, and the number challenging in any one firm falls except at the critical values of $s$ beyond which one less firm is challenged (and those who were challenging this firm must reallocate across other firms). Finally, Proposition 3.6 restates Lemma 5; type 2 shareholders do better in PSSPEs in which more type 1 shareholders are challenged.

To this point, we have considered only two sizes of block shareholders, which has enabled us to obtain an analytic characterization of equilibria. However, results analogous to Propositions 1 and 2 can be obtained without such an assumption. The following
result is a straightforward generalization of these two propositions when the larger type 1 shareholders can take on different sizes.

**Proposition 4.** Suppose there are $N$ type 1 shareholders with wealth $x_i$, and $M$ type 2 shareholders with wealth $y$, where $x_i > y$, for $i = 1, 2, \ldots, N$. Then,

A. The following characterizes a set of PSSPEs. All type 1 shareholders with positions smaller than a cut-off level $\bar{x}$ are challenged, and all those with positions greater than $\bar{x}$ are not. Type 1 shareholders smaller than or equal to $\bar{x}$ put all their wealth in one firm. Type 1 shareholders larger than $\bar{x}$ put $\bar{x}$ in their first firm, and then stake out another firm with remaining funds (if this is still greater than $\bar{x}$, they stake out yet another firm, etc.).

Type 2 shareholders will allocate themselves such that benefits they receive are equalized across all firms in which they invest. The larger the type 1 shareholder in any firm, the fewer type 2 shareholders will challenge that firm; type 1 shareholders beyond $\bar{x}$ are not challenged.

B. While many such PSSPEs (corresponding to different cut-off levels) may exist, the equilibrium in this class with the maximal cutoff level is the unique CSE and PCPNE.

Figure 4 depicts the distribution of type 2 shareholders in such an equilibrium. Higher benefit curves correspond to firms with smaller type 1 shareholders. In this diagram, type 1 shareholders are challenged if and only if $s \leq 5$. A shareholder of size $s = 5$ is challenged by the number of type 2 shareholders which maximizes benefits per challenger against such a shareholder.

The CSE of this proposition is similar to that of Proposition 2, with the added effects that the number of challengers a type 1 shareholder faces decreases with the size of his position, and that type 1 shareholders are not challenged at all if their positions are larger than a threshold level $\bar{x}$. Similar results could likewise be obtained under further
generalization allowing type 2 shareholders to take on arbitrary sizes smaller than the type 1 shareholders.

It is important to note that while the use of Shapley values led to the simple analytic characterizations for the division of control benefits employed in this section, results do not qualitatively depend on this specific formulation. General conditions on how control benefits are divided within a firm that are sufficient for Propositions 1, 2, and 4 to hold are stated in Appendix B. All of these assumptions are quite plausible, particularly if one is willing to accept an underlying cooperative basis for the division of control benefits. Consequently, these conditions serve both to reinforce the generality of our results and to highlight what aspects of Shapley values are important for these results.

4. EXAMPLES AND EMPIRICAL EVIDENCE

In this section we consider several simple numerical examples to give an intuitive feel for what the above equilibria prescribe, and then present some preliminary empirical evidence that is consistent with our theory. In considering the examples, it is worth recalling Proposition 3.3, which indicates that identical equilibria will be obtained if parameters are proportionally scaled up or down.

Example 1. Consider an economy with 3 firms, 2 type I block shareholders, and 60 type 2 block shareholders, where type 1 shareholders are 5 times as large as type 2 shareholders. In all PSSPE equilibria, each of the two block shareholders allocates all wealth to separate firms (which we will designate firms 1 and 2 without loss of generality). To within integer roundings, there are three PSSPE given by the following configurations for type 2 shareholders:

1. 60 type 2 shareholders in firm 3 (neither type 1 shareholder is challenged). Expected benefits are given by $\phi_1 = 1$, $\phi_2 = 0.017$.
2. 27.1 type 2 shareholders in firm 1 or 2 and 32.9 in firm 3 (one type 1 shareholder is challenged). Expected benefits are $\phi_1 = 0.589$, $\phi_2 = 0.031$.
3. 17.9 type 2 shareholders in both firms 1 and 2 and 24.2 in firm 3 (both type 1 shareholders are challenged). Expected benefits are $\phi_1 = 0.265$, $\phi_2 = 0.041$.

The last of these equilibria is the CSE equilibrium, and is the best for type 2 shareholders.

If the number of type 2 shareholders $M$ increases to 80, the number of such shareholders challenging each type 1 shareholder in the CSE increases to 24.7, with 30.6 such shareholders in firm 3. Expected benefits in turn fall to $\phi_1 = 0.195$, $\phi_2 = 0.033$.

Instead letting $s$ increase to 10 (for the initial case where $M = 60$), there are only 2 PSSPE, given by integer rounding of,

1. 60 type 2 shareholders in firm 3.
2. 21.4 type 2 shareholders in firm 1 or 2 and 38.6 in firm 3.

20. However, see Beja and Gilboa (1990) for one example in which these assumptions are not satisfied. In their setting, large individuals may be left out of coalitions in the initial stage of multi-stage cooperative games because of their ability to dominate a coalition in a latter stage.

21. That is, in a PSSPE, the number of type 2 shareholders allocated across firms is given by integer values differing by less than one from the stated values (which are the levels that equate benefits for such shareholders across all firms).
Example 1. Consider an example where there are 500 firms, 200 large block shareholders and 10,000 smaller block shareholders. (Recall from Section 3 that these are the parameters depicted in Figures 2 and 3 when $s = 5$ and $s = 8$ respectively.) Table 1 characterizes PSSPE for this case. In particular, the first column of this table identifies a particular PSSPE by giving the number of type 1 shareholders $J'$ challenged. The second column gives the critical levels $s_{J'}$ of Proposition 1, beyond which $J'$ shareholders can no longer be challenged in a PSSPE. These values fall within a reasonable range; if $s < 6.92$, all type 1 shareholders can be challenged in equilibrium, while none can be challenged if $s > 9.0$. Following columns give the number of type 2 shareholders challenging each type 1 shareholder, benefits of type 1 and type 2 shareholders, and the value of concentration in different PSSPE equilibria with $J'$ firms challenged, when $s = 5$ and $s = 8$. The value of concentration is defined as the ratio $\phi_1/(\phi_2 s)$, the premium in benefits per share that type 1 shareholders realize over type 2 shareholders due to their concentration.

Note that as the number of firms $J'$ challenged increases, $M_2$, $\phi_1$, and the value of concentration decreases, and $\phi_2$ increases. Comparing the $s = 5$ and $s = 8$ case for a fixed
number of challenged firms $J'$, $M_2$ and $\phi_2$ are smaller and $\phi_1$ is larger for the higher level of $s$. When $s=5$, all 200 firms can be challenged in a PSSPE, while instead, when $s=8$ (the type 1 shareholders are relatively larger), at most 71 type 1 shareholders can be challenged in a PSSPE. Graphically, when $s=5$ (Figure 2), all 200 firms can be challenged without $M_2$ reaching the peak of the type 2 shareholders benefits curve, while when $s=8$ (Figure 3), the peak is obtained when $s=71$. Proposition 2 indicates that it is these PSSPE equilibria, with the maximal number of firms challenged, that are the unique CSE for $s=5$ and $s=8$ respectively. Note that in moving from the CSE with $s=5$ to that with $s=8$, type 1 shareholders gain more from being less likely to be challenged than from the direct effect of obtaining a greater share of benefits when challenged.

Empirical evidence

While preliminary in nature, the following evidence corresponds with the predictions of this paper. For the 456 firms reported in 1981 CDE Stock Ownership Directory: Fortune 500, there are 246 shareholders holding blocks of greater than 10\% of a firm's equity and 123 holding blocks greater than 20\%. Table 2 gives both the actual distribution across firms of these block shareholders and the expected distribution under a random allocation. This table supports the hypothesis that larger shareholders indeed do "create their own space". For both the 10\% and 20\% blocks, there are many fewer firms with multiple large-block shareholders and many more firms with a single large-block shareholder than a random distribution would predict. For example, while a random distribution would predict approximately 14 firms with two or more shareholders holding 20\% blocks, only 3 such

22. For the random allocation, each large block is taken to have an identical chance of being located in any firm. While this could conceivably violate an allocation constraint, i.e. more than 100\% of the shares in a firm are held, the possibility is remote enough to be inconsequential.
firms exist. A goodness of fit chi-square test finds the difference between the actual and random distribution significant at 0.001 and 0.003 levels, for the 10% and 20% blocks respectively.\(^{23}\)

Similarly, there is empirical support for a clientele effect in the shareholder structure. Regressing the number of 1% block shareholders on the size of the largest block shareholder and a constant, one finds a negative coefficient for the size of the largest block, significant at a 0.001 level.\(^{24}\)

Additionally, several other papers have found indirect evidence supporting the existence of a threshold level. Barclay and Holderness (1989) finds that blocks whose size is on the order of 25% of a firm’s equity sell for a significant premium. Morck, Shleifer and Vishny (1988) find the presence of such large blocks affects managerial performance. Both these papers discuss the possibility of a threshold level.

5. CONCLUSION

Under the assumption of divisible control benefits, this paper develops a model of corporate control which both justifies block investment and yields rich new implications for the shareholder structure across firms. In particular, the model predicts that large block shareholders will “create their own space”, in the sense that their presence in a firm will deter other block investors; the shareholder structure of block investors in corporations will exhibit a clientele effect; and there will be a threshold size beyond which large investors will not be challenged. All these predictions are supported by a preliminary review of empirical evidence. A more thorough empirical examination on the distributional structure of block shareholders and on dynamic changes in this structure (in particular, how a change in the size of the leading shareholder affects the size of other block shareholders) would be very interesting.

Many further empirical implications which could serve to distinguish this model from other stories of block shareholding follow from relaxing several assumptions. Allowing for firms of different sizes or with different control benefits would imply that firms with greater benefits per unit size will attract larger leading block shareholders. This could be tested to the extent that one can exogenously identify firms or industries with high private benefits.\(^{25}\) Similarly, if a significant fraction of private benefits are derived from selling blocks at premiums to control contestants, one would also expect to see greater shareholder concentration in industries undergoing such contests. This theory also predicts that in firms with multiple classes of equity, ownership concentration will be greater in classes with superior voting privileges.

While this model captures many aspects of block shareholding, it ignores elements important to any complete theory of corporate control. Most significantly, there is little role for management. In the model, management is taken to be an extension of a supporting

\(^{23}\) An examination of the data shows a size effect as well; there are more large blocks in small firms than large firms. This suggests that we should find more firms with multiple large blocks than under the random distribution, making the observed effect more striking.

\(^{24}\) Nor is this result driven by an allocation constraint, whereby some firms with large leading shareholders have almost all shares held by block shareholders. The typical firm in the sample has only about 50% of its shares held by shareholders of size 1% or greater, leaving plenty of shares available for another investor desiring to hold such a block. Furthermore, regressing the number of 1% blocks normalized by the fraction of shares not held by the largest shareholder on the size of the largest shareholder—a fairly stringent test—still yields a negative coefficient; however, this is only significant at a 0.1 level.

\(^{25}\) Demsetz and Lehn propose that professional sports teams and the communications industry are likely to yield high private benefits to control; and these indeed are two industries with high leading ownership concentration.
coalition of shareholders, while realistically, agency problems and managerial entrenchment are likely to play an important role in many control contests. This model could be extended to consider management as a participant in the control game, where management receives benefits either due to support from their own private holdings, noise traders, firm controlled funds, or institutional entrenchment.

Along similar lines, allowing for the shareholder structure to affect public as well as private benefits would allow one to explore consequences of the interaction between competitive shareholder activity (over the division of private benefits) and cooperative activity (such as providing the public good of monitoring management). For example, the presence of a large shareholder might be more conducive to monitoring or disciplining activity, thereby increasing public benefits. This in turn likely to yield interesting implications for corporate decisions, such as capital structure decisions. In particular, by altering capital structure, a firm could affect its equity/private benefits ratio, and thereby change its equilibrium shareholder structure and consequently its market value. While such corporate policy considerations are beyond the scope of this paper’s analysis (which holds public benefits fixed), the framework developed here seems well suited to analyse these and other related questions.

APPENDIX A

Proof of Lemma 2. From Lemma 1, we can consider a majority voting game with one individual of size $s$, $L$ individuals of size 1, and noise distributed as $U[-1/2, 1/2]$. First suppose that liquidity traders do not vote. Let $s \leq L + 1$. The Shapley value for the agent of size $s$ is given by,

$$
\phi_1 = \frac{1}{L+1} \sum_{i=0}^{L} [\alpha \geq i > \alpha - s],
$$

where $\alpha$ is an indicator function and $\alpha = (s + L)/2$ is the vote needed for a majority; half of all votes cast. Equation (A1) states that the benefits of agent 1 are given by the probability that he will be the swing vote in a random ordering of shareholders. Re-arranging (A1) yields,

$$
\phi_1 = \frac{1}{L+1} \left( \text{int} (s) + 1 \mod (s + L)/2 \right),
$$

(A2)

where int (·) and mod (·) give the nearest lower integer and the modulus one of their respective arguments.

Now consider the effect of the noise votes. The number of votes $\alpha$ needed for a majority increases by $\beta$, while the number of votes in individual 1's coalition increases by $\beta + \bar{\epsilon}$. Equation (A2) therefore becomes,

$$
\phi_1 = \frac{1}{L+1} E(\text{int} (s) + 1 \mod (s + L)/2 - \bar{\epsilon)),
$$

(A3)

Since $\bar{\epsilon}$ is distributed as $U[-1/2, 1/2]$, $\mod ((s + L)/2 - \bar{\epsilon})$ is distributed as $U[0, 1]$. Thus,

$$
\phi_1 = \frac{1}{L+1} (\text{int} (s) + \mod (s)) = \frac{s}{L+1}.
$$

(A4)

And since $\phi_1 + L\phi_2 = 1$,

$$
\phi_2 = \frac{L+1-s}{L(L+1)}.
$$

26. Harris and Raviv (1988a) and Stulz (1988) examine the effect on firm value that capital structure may have by affecting the size of the controlling shareholder’s block and consequent control contests. These papers thus consider the effect of capital structure in a partial equilibrium framework where a single large shareholder has only one firm in which to invest. The present framework allows for the examination of potentially richer effects that capital structure may have on the overall shareholder structure, in a setting where firms must compete for “desirable” shareholder structures.
If instead \( s > L + 1 \), individual 1 wins all votes for any realization of \( \tilde{e} \), and therefore \( \phi_1 = 1 \) and \( \phi_2 = 0 \).

**Proof of Lemma 3.** From Lemma 1, we can consider a majority voting game with one individual of size \( s_0 \), one of size \( s_1 \), \( L \) individuals of size 1, and noise distributed as \( U[-1/2, 1/2] \). First, suppose there is no noise, and let \( s_0 \leq L + s_1 + 1, s_1 \leq L + s_0 + 1 \). The Shapley value of agent 1 (of size \( s_1 \)) is given by,

\[
\phi_1 = \frac{1}{L + 2} \left[ \sum_{i=0}^{L+1} \left( \frac{i}{L+1} \phi_1(1, i) + \frac{L+1-i}{L+1} \phi_1(2, i) \right) \right],
\]

where \( \phi_1(1, i) \) and \( \phi_1(2, i) \) are the payoffs to agent 1 for appearing in the \( i \)th position of a random ordering of all block shareholders from 0 to \( L + 1 \), respectively after/before agent 0 has appeared. The weights are the probabilities of such an ordering. \( \phi_1(1, i) \) and \( \phi_1(2, i) \) equal one in their respective cases if agent 1 is the swing voter, otherwise they are 0. That is,

\[
\phi_1(1, i) = I[(i-1) + s_0 \leq \alpha < (i-1) + s_0 + s_1]
\]

\[
\phi_1(2, i) = I[i \leq \alpha < i + s_1];
\]

where \( \alpha \), half of all votes cast, is given by \( (L + s_1 + s_2)/2 \), and \( I \) is the indicator function. Re-arranging terms yields,

\[
\phi_1(1, i) = I\left[ \frac{L - s_0 + s_1 + 2}{2} \geq i \geq \frac{L - s_0 - s_1 + 2}{2} \right]
\]

\[
\phi_1(2, i) = I\left[ \frac{L + s_0 + s_1}{2} \geq i \geq \frac{L + s_0 - s_1}{2} \right].
\]

Then noting that the contributions from when agent 0 appears before and after agent 1 are symmetric,

\[
\phi_1 = \frac{2}{L + 2} \left[ \sum_{i=0}^{L+1} \left( \frac{i}{L+1} \phi_1(1, i) + \frac{L+1-i}{L+1} \phi_1(2, i) \right) \right],
\]

And now adding noise, expected benefits are given by,

\[
\phi_1 = \frac{2}{(L + 1)(L + 2)} \mathbb{E} \left[ \sum_{i=0}^{L+1} I \left[ \frac{L - s_0 + s_1 + 2}{2} \geq i \geq \frac{L - s_0 - s_1 + 2}{2} - \tilde{e} \right] \right].
\]

Define the random variables,

\[
\tilde{u}_i \equiv \text{int} \left[ \frac{L - s_0 + s_1 + 2}{2} - \tilde{e} \right] ; \quad \tilde{y}_i \equiv \text{int} \left[ \frac{L - s_0 - s_1 + 2}{2} - \tilde{e} \right].
\]

\( \tilde{u}_i \) and \( \tilde{y}_i \) have means \((L - s_0 + s_1 + 1)/2\) and \((L - s_0 - s_1 + 1)/2\) respectively. Define \( \bar{u}_i = \text{mod} \left[ (L - s_0 + s_1)/2 \right] \), and \( \bar{y}_i = \text{mod} \left[ (L - s_0 - s_1)/2 \right] \). The variances of \( \tilde{u}_i \) and \( \tilde{y}_i \) are given by,

\[
\sigma^2_{\tilde{u}_i} = \left\{ \begin{array}{ll}
(\bar{u}_i + 1/2)(1/2 - \bar{u}_i) & \text{if } \bar{u}_i \leq 1/2 \\
(\bar{u}_i - 1/2)(3/2 - \bar{u}_i) & \text{if } \bar{u}_i > 1/2
\end{array} \right.
\]

\[
\sigma^2_{\tilde{y}_i} = \left\{ \begin{array}{ll}
(\bar{y}_i + 1/2)(1/2 - \bar{y}_i) & \text{if } \bar{y}_i \leq 1/2 \\
(\bar{y}_i - 1/2)(3/2 - \bar{y}_i) & \text{if } \bar{y}_i > 1/2
\end{array} \right.
\]

which can be expressed as,

\[
\sigma^2_{\tilde{u}_i} = |\bar{u}_i - 1/2|(1 - |\bar{u}_i - 1/2|); \quad \sigma^2_{\tilde{y}_i} = |\bar{y}_i - 1/2|(1 - |\bar{y}_i - 1/2|).
\]
Now if $s_0 + s_1 \leq L + 1$, then for all realizations of $\varepsilon, \bar{u}_i \geq 0$ and $\bar{u}_i \leq L + 1$, and therefore equation (A9) yields the weighted sum from $y_1 + 1$ to $y_i$, given by,

$$\phi_i = \frac{2}{(L+1)(L+2)} E[\sum_{i=1}^{n} l_i] = \frac{1}{(L+1)(L+2)} E[\bar{u}_i(y_i + 1) - y_i(y_i + 1)]$$

$$= \frac{1}{(L+1)(L+2)} \left\{ \left[ \frac{(L - s_0 + s_1 + 1)^2}{2} + \frac{(L - s_0 + s_1 + 1)}{2} \sigma_{u_i}^2 \right] + \left[ \frac{(L - s_0 - s_1 + 1)^2}{2} + \frac{(L - s_0 - s_1 + 1)}{2} \sigma_{u_i}^2 \right] \right\}$$

$$= \frac{s_1(L - s_0 + 2) + (\sigma_{u_i}^2 - \sigma_{u_i}^2)}{(L+1)(L+2)}$$

(A13)

If instead $s_0 + s_1 > L + 1$, then for all realizations of $\varepsilon, y_1 \leq 1$, and $s_1 \leq L + 1 + s_0$ ensures that $\bar{u}_i \leq L + 1$. Therefore, equation (A9) yields,

$$\phi_i = \frac{2}{(L+1)(L+2)} E[\sum_{i=0}^{n} l_i] = \frac{1}{(L+1)(L+2)} E[\bar{u}_i(y_i + 1)]$$

$$= \frac{1}{(L+1)(L+2)} \left\{ \left[ \frac{(L - s_0 + s_1 + 1)^2}{2} + \frac{(L - s_0 + s_1 + 1)}{2} \sigma_{u_i}^2 \right] \right\}$$

$$= \frac{(L - s_0 + s_1 + 2)^2 + 4\sigma_{u_i}^2 - 1}{4(L+1)(L+2)}$$

(A14)

Finally, if $s_1 > L + s_0 + 1$, then individual 1 wins all votes for any realization of $\varepsilon$, and therefore $\phi_1 = 1, \phi_0 = 0$; and if $s_0 > L + s_1 + 1$, individual 0 wins all such votes and therefore $\phi_1 = 0, \phi_0 = 1$. Of course, the derivation for $\phi_0$ is identical.

**Proof of Proposition 1.** First we consider the play of type 2 shareholders. Within integer deviations, in equilibrium $\phi_2$ must be equated across all firms with type 2 shareholders, and be decreasing in $L(j)$, so that no type 2 shareholders want to change firms. Presuming no diversification (as shown below), this implies that given equilibrium play of type 1 shareholders, equations (4) through (7) must hold. From Lemma 2 and equation (4), $\phi_2$ is given by,

$$\phi_2 = \frac{M_2 + 1 - s}{M_2(M_2 + 1)} \forall j \in J'$$

(A15)

$$\phi_2 = \frac{1}{M_1} \forall j \in J \setminus J(N).$$

(A16)

Equating these values, imposing equation (5), and solving for $M_2$ yields,

$$M_2 = \frac{(M + J's - (J - N + J'))^2 + 4M(J - N + J')(1 - s)}{2(J - N + J')}$$

(Equations (5)-(7) are sufficient conditions to characterize equilibrium play of type 2 shareholders. Since equation (A17) follows from equations (5) and (6), it follows that a root of this equation characterizes an equilibrium provided that (7) is satisfied as well. Using (A15), this will be true provided that,

$$M_2(s) \geq (s - 1) + \sqrt{s(s - 1)}.$$

(A18)

This condition is never satisfied at the negative root of (A17) (see the text and Figure 1 for intuition), as $\frac{\partial \phi_2}{\partial M_2} > 0$ for firms with a type 1 investor (thereby inducing type 2 investors not in such firms to switch to them). At the positive root, it is easy to see equation (A18) is satisfied when $s = 1$. Furthermore, this positive root decreases with $s$, while the right-hand side of equation (A18) increases unboundedly with $s$. Thus for a given $J'$, there exists a unique value $s_J$ such that the positive root of (A18) denotes an equilibrium if and only if $s \leq s_J$. Furthermore, since this root decreases with $J'$, it follows that $s_J$ decreases with $J'$ as well. 27

27. When $s$ is sufficiently large, the radical in (A17) becomes negative, at which point no values of $M_1$ and $M_2$ satisfy equations (5) and (6). One can show that for all $J'$, this occurs at a value $s_J$ exceeding $s_J$, and thus is not a binding constraint in restricting equilibria.
To complete the proof, it must be shown that both type 1 and type 2 shareholders don't want to deviate by diversifying, and that type 1 shareholders don't deviate by jointly investing in the same firm. The desired result for type 1 shareholders can easily be shown, along the lines of footnote 18, by considering off-the-equilibrium-path play of type 2 shareholders in which they play the worst continuation equilibrium for a type 1 deviator. However, we put off showing type 1 shareholders don't deviate until the proof of Proposition 2, where we show the stronger result that they will not deviate even when we restrict type 2 shareholders to responding by challenging the maximal number \( k \) of type 1 shareholders which they can challenge in a PSSPE.

We use Lemmas 2 and 3 to show that diversification is unprofitable for a type 2 agent. In equilibrium, a type 2 shareholder will receive benefits of,

\[
\phi^*_2 = \frac{1}{M_1} = \frac{M_2 + 1 - s}{M_2(M_2 + 1)}.
\]  
(A19)

In order to show the desired result, it is sufficient to show that by deviating and holding \( m_j < m \) in any firm \( j \), investor \( i \) obtains benefits of less than or equal to \( (m_j/m)\phi^*_2 \) in this firm. Suppose first that \( j \in J \setminus J(N) \). Then, since this firm will consist of \( M_1 \) individuals of size \( m \) and one of size \( m_j \), Lemma 2 implies that,

\[
\phi^*_i = \left( \frac{m_j}{m} \right) \left( \frac{M_2 - s + 2}{M_2 + 2} \right) \left( \frac{M_1 + 1}{M_1} \right) = (m_j/m)\phi^*_2.
\]  
(A20)

If instead \( j \in J \setminus J(N) \), then \( j \) will consist of one individual of size \( n = sm \) (the type 1 investor), one of size \( m_j \) and \( M_2 \) of size \( m \). From Lemma 3,

\[
\phi^*_i = \left( \frac{m_j}{m} \right) \left( \frac{M_2 - s + 2}{M_2 + 1} \right) \left( \frac{M_1 + 1}{M_1} \right) = (m_j/m)\phi^*_2.
\]  
(A21)

where the middle inequality follows from equations (A12) and (A18).

Proof of Lemma 4. Define \( M^*_2 = \arg \max_i \left( \phi^*_i(L), j \in J(N) \right) \) and \( \phi^*_2 = \max_j \left( \phi^*_i(L), j \in J(N) \right) \), and let \( M_{2,J} \) denote the positive root of \( M_2 \) in (A17) for any real value of \( J \). There is a PSSPE with \( J \) type 1 shareholders challenged if and only if equation (7) is satisfied at \( M_{2,J} \). Equation (7) in turn is satisfied if and only if \( M_{2,J} \geq M^*_2 \), since \( \phi^*_2 \) is single-peaked at \( M^*_2 \) for \( j \in J \). Thus, if \( k(s) \neq N, M_{2,J} \leq M^*_2 \), because by definition \( k(s) \) is the maximal number of type 1 shareholders that can be challenged in equilibrium.

Now suppose \( k(s) < N \). Differentiating equation (A17) demonstrates that \( M_{2,J} \) is decreasing in \( J \). (This is shown below in Lemma 5 for \( J \leq k(s) \).) Hence, \( M_{2,J} = M^*_2 \) and \( \phi^*_i(M_{2,J}) = \phi^*_i \) for some \( z \) between \( k(s) \) and \( k(s) + 1 \). That is, within integer roundings on \( k, M_{2,J} = \arg \max_i \left( \phi^*_i(L), j \in J(N) \right) \).

Proof of Lemma 5. Equations (5), (6) and (7) hold for all equilibria. It follows that sign \( \partial M_1/\partial J \) = sign \( \partial M_2/\partial J \). Differentiating equation (5) with respect to \( J \) yields,

\[
\frac{\partial M_1}{\partial J} (J - N) + \frac{\partial M_2}{\partial J} J + M_2 = 0.
\]  
(A22)

\( J \geq 0, J - N \geq 0, \) and \( M_2 > 0 \) imply that the sign of the partials must be negative. And since \( \phi_2 = 1/M_1(J) \), it follows that \( \partial \phi_2/\partial J > 0 \). 

Proof of Proposition 2. First we show that Proposition 2 specifies CSE strategies for type 2 shareholders, and then we show type 1 shareholders play as specified given the strategies of type 2 shareholders.

Lemma 5 indicates that the equilibrium with the maximal number of type 1 shareholders challenged dominates all other equilibria for type 2 shareholders. Hence, if it is class strong, it is the unique CSE; all other PSSPE are not immune to a joint deviation by all type 2 shareholders to this superior equilibrium. Now suppose the equilibrium with the maximum number of firms \( k \) challenged is not class strong. Then there exists some coalition of type 2 shareholders that can Pareto improve by deviating. Any deviation involves an increase in the number of type 2 shareholders in some firm. Denote such a firm \( j \). Now since in equilibrium, \( \partial \phi^*_2/\partial L < 0 \forall j \in J \setminus J(N), j \in J \), and \( \phi^*_i(L) \) is single-peaked for \( j \in J \) and strictly decreasing for \( j \notin J \setminus J(N) \), it follows that any increase in the number of type 2 shareholders in these firms decreases their benefits. Hence it must follow that \( j \in J \setminus J(N) \). That is, a Pareto improvement among a coalition must involve challenging at least one type 1 shareholder not challenged in the proposed equilibrium. But by assumption, the maximal equilibrium number of firms \( k \) are challenged. Lemma 4 indicates that this implies either that all \( N \) type 1 shareholders are

28. The dependence of \( M^*_2 \) and \( \phi^*_2 \) on \( s \) is notationally suppressed in this proof, where \( s \) is fixed.
already challenged, or that the number of type 2 shareholders challenging type 1 shareholders is \( M^*_2 \) (the number which maximizes possible benefits per challenger when challenging a type 1 shareholder). In either case, it is not possible to challenge another firm and increase benefits. Hence, deviators who move to firm \( j' \) do not improve over the equilibrium. Thus, the equilibrium is a class strong equilibrium.

Class Strong Equilibrium implies that it is perfect coalition proof (since the set of potential equilibrium-breaking deviations allowable in the former contain the set allowable in the latter). Furthermore, since a deviation by all type 2 shareholders from any other PSSPE to the Class Strong Equilibrium is a Pareto improvement, which is in turn immune to counter-deviations from within this set (since it is class strong), the stated equilibrium is the unique PCPNE.

It remains to be shown that type 1 shareholders don't diversify given that type 2 shareholders are playing CSE strategies. Insofar as this proof is notationally somewhat messy, it may be useful to first briefly state in words the direction of the proof. First we show that when individual \( i \) splits her wealth, it becomes easier for type 2 shareholders to challenge these two firms than one in which \( i \) has all her wealth and one in which there are no type 1 shareholders. This implies that the same number of type 2 shareholders in these two firms can obtain greater benefits if \( i \) splits her wealth. And this in turn induces more type 2 shareholders into these firms, weakly raising their benefits relative to individual \( i \) not diversifying. With more type 2 shareholders challenging these two firms and their benefits at least as high as before the deviation, individual \( i \) must be doing worse.

Suppose some type 1 shareholder \( i \) deviates by holding positions in two firms in equilibrium (the proof for greater than two firms follows by induction). Call these firms \( j_1 \) and \( j_2 \), and the shares \( s_1 \) holds in these firms normalized by \( m, s'_1, \) and \( s'_2 \), where \( s'_1 + s'_2 = s \). In the CSE, type 2 shareholders respond by challenging both these firms (as these positions will be smaller than other type 1 shareholder positions \( s \)). We will show that if \( i \) are higher in the initial equilibrium than under this deviation. It is obviously sufficient to show this is the case even when \( i \) is one of the shareholders challenged by type 2 shareholders in the initial equilibrium (the toughest case).

Some notation is needed. Let \( \phi_2 \) and \( \phi_3 \) denote control benefits of type 2 shareholders in respectively the initial equilibrium and under the deviation. Let \( L(x) \) and \( \bar{L}(x) \) be the number of type 2 shareholders in a firm with a type 1 shareholder of size \( x \) in the initial equilibrium and after the proposed deviation. \( L(x) \) is defined for \( x \in [0, s] \) and \( \bar{L}(x) \) is defined for \( x \in \{0, s_1, s_2, s\} \). Extend \( L(x) \) to \([0, s] \) by defining \( L(x) \) as the positive root of the solution to,

\[
\phi_2 = \frac{1}{L(0)} = \frac{L(s) + 1 - x}{L(x)(L(x) + 1)},
\]

(A23)
similarly extend \( \bar{L}(x) \) to \([0, s] \) replacing \( \phi_2 \) and \( L(0) \) above with \( \bar{\phi}_3 \) and \( \bar{L}(0) \). It is clear from Lemma 2 that \( L(x) \) and \( \bar{L}(x) \) give the number of type 2 shareholders in a firm with a type 1 shareholder of the size \( x \) such that type 2 shareholders obtain control benefits given by \( \phi_2 \) or \( \bar{\phi}_3 \). Furthermore, \( L(x) \) and \( \bar{L}(x) \) are continuous on \([0, s] \).

Solving for \( L(x) \) in (A23) yields,

\[
L(x) = \frac{-(L(0) - 1) + \sqrt{(L(0) - 1)^2 - 4(x - 1)L(0)}}{2L(0)},
\]

(A24)

Thus,

\[
L''(x) = -2L(0)[(L(0) - 1)^2 - 4(x - 1)L(0)]^{-3/2} < 0.
\]

(A25)

Therefore \( L(x) \) is concave over \([0, s] \), and hence, \( \forall s_1, s_2 \) such that \( s_1 + s_2 = s, s_1, s_2 > 0 \),

\[
L(s_1) + L(s_2) > L(0) + L(s).
\]

(A26)

Now suppose first that type 1 shareholders are not all challenged in the proposed equilibrium. Then from Lemma 4, the number of type 2 shareholders who challenge a type 1 shareholder is \( M^*_2 = \arg\max_x \phi_2(L) \), and \( \phi_2 = \arg\max_x \phi_2(L) \). Similarly, a result analogous to Lemma 4 holds when type 1 shareholders have shareholdings of different sizes (as under the deviation). In particular, in the equilibrium where type 2 shareholders are challenging the maximal number of firms possible, the largest type 1 shareholder actually contested is challenged by the number of type 2 shareholders that maximizes their individual benefits versus a shareholder of this size. Since in this case, provided that more than two firms are challenged, \( 29 \) the largest challenged shareholder will

29. If not, the proof follows along the following lines. By following the prescribed equilibrium a type 1 shareholder is very unlikely to be challenged, while if this shareholder deviates by diversifying, this shareholder will hold the smallest positions of any type 1 shareholder and will be challenged in all of them. This deviation can then be shown to be worse than the proposed equilibrium.
once again be of size \( s \), once again \( \phi_2 = \max \phi_i(L) \). Hence,
\[
\phi_2 = \phi_i; \quad \text{and therefore,} \quad L(s_1) = L(s_1), \quad L(s_2) = L(s_2); \tag{A27}
\]
and hence, from (A26) it follows that,
\[
\bar{L}(s_1) + \bar{L}(s_2) > L(0) + L(s). \tag{A28}
\]

Benefits in both the proposed equilibrium and under the deviation for shareholder \( i \) are given by the total benefits from the two firms in which \( i \) invests net the benefits type 2 shareholders receive from these firms. Hence, defining \( \phi_{i,j} \) and \( \phi_{i,j}^{*} \) as the benefits for the deviating type 1 individual \( i \) under the proposed equilibrium and the deviation, it follows that,
\[
\phi_{i,j} = 1 - (L(s))\phi_2 = 2 - (L(0) + L(s))\phi_2 \tag{A29}
\]
\[
\tilde{\phi}_{i,j} = 2 - (\bar{L}(s_1) + \bar{L}(s_2))\phi_2. \tag{A30}
\]
The second equality in (A29) is obtained by noting that type 2 shareholders receive all benefits in any firm without a type 1 shareholder. Equations (A27) through (A30) together imply that \( \phi_{i,j} > \tilde{\phi}_{i,j}^{*} \), which was to be shown.

The proof is similar when instead the initial equilibrium is such that all type 1 shareholders are challenged. Only a sketch is given here for brevity. In this case, it can once again be shown that (A28) holds. Intuitively this follows from (A26) which indicates that benefits of \( \phi_2 \) can be obtained by more type 2 shareholders in firms \( j_1 \) and \( j_2 \) than before the deviation, and therefore greater benefits are obtained by the same number of type 2 shareholders in these firms. This implies that the number of type 2 shareholders in these firms will increase (hence (A28)), the number in all other firms will decrease, and therefore \( \tilde{\phi}_2 > \phi_2 \). Then once again, (A28), (A29), (A30) and \( \phi_2 > \phi_2 \) together imply that \( \phi_{i,j} > \phi_{i,j}^{*} \). \( \square \)

Proof of Proposition 4 (Sketch). The proof that Part A characterizes PSSPEs is essentially identical to that of Proposition 1 and therefore is not presented here. For Part B, first suppose there does not exist a cut-off level \( \hat{s} \). Then there exists two type 1 investors \( i, i' \), \( x_i > x_i' \), with \( i \) challenged and \( i' \) not challenged. But this cannot be Class Strong, because the collection of type 2 shareholders challenging \( i \) could affect a Pareto improvement by challenging \( i' \) instead. Thus any CSE must involve a cut-off level.

The higher the cut-off level, the more firms are challenged. Suppose cut-off levels of both \( \hat{s}_1 \) and \( \hat{s}_2 \) yield PSSPE, with \( \hat{s}_1 > \hat{s}_2 \). Challenging more firms in equilibrium implies there are less type 2 shareholders challenging in each firm, and therefore benefits for type 2 shareholders are greater. Hence, only the \( \hat{s}_1 \) equilibrium can be a CSE, as the \( \hat{s}_2 \) equilibrium is not immune to a deviation by all type 2 shareholders to the \( \hat{s}_1 \) equilibrium.

To show the equilibrium with the maximal cut-off level is class strong, suppose there exists some Pareto-improving deviation among type 2 shareholders. Just as in the proof of Proposition 2, in any PSSPE \( \partial \phi_i(L)/\partial L < 0 \) for all firms with type 2 shareholders, this deviation must involve challenging some type 1 shareholder not previously challenged. However, in a manner analogous to Lemma 4, one can show that in the equilibrium with the maximal cutoff level, either all type 1 shareholders are challenged, or the largest type 1 shareholder actually challenged is challenged by the number of shareholders that maximizes possible benefits per challenger against such a shareholder. (If this were not true, a higher cut-off level in which one more type 1 shareholder was challenged would be a PSSPE.) This implies that in the prescribed equilibrium, benefits for type 2 shareholders are greater than could ever be obtained from challenging a larger (previously unchallenged) shareholder. Hence, such a deviation cannot be Pareto-improving. CSE implies this is also a PCPNE, and uniqueness follows in the same manner as in Proposition 2.

Type 1 shareholders smaller than \( \hat{s} \) don't want to diversify for precisely the same reason as in Proposition 2. And type 1 shareholders larger than the cut-off level want to keep only enough in their first firm to ensure they won't be challenged, and invest the remainder elsewhere. \( \square \)

30. The proof that two different type 1 shareholders don't hold blocks in the same firm is almost identical to the proof that they don't diversify; details are omitted for brevity. Basically, one shows that a fixed number of type 2 shareholders can do better in two firms if one contains two large block shareholders and the other none, than when each firm contains one large block shareholder. Thus, a pair of firms with two large investors will attract more type 2 shareholders, who will also be weakly better off, if the large investors jointly invest in one of the firms than if they create individual stakes in the two firms. This in turn implies that these two type 1 shareholders will be worse off, thereby ruling out such a deviation.
APPENDIX B—SUFFICIENT CONDITIONS ON THE DIVISION OF SURPLUS

The following conditions on the manner in which private benefits within a firm are divided among block shareholders are sufficient for Propositions 1, 2 and 4 to hold. Zwiebel (1991) gives proofs, which follow along the lines of those for Section 3 (albeit, with considerably more notational complexity), and also discusses and interprets these assumptions.

A1. Symmetry. For any permutation \( \pi \) of the \( I \) shareholders in a firm, \( \forall i, \phi(s_1, \ldots, s_i, \ldots, s_I) = \phi_{\pi(i)}(s_{\pi(1)}, \ldots, s_{\pi(I)}, \ldots, s_{\pi(I)}) \).

A2. Individual Rationality and Optimality. \( \phi_i \geq 0, \sum \phi_i = 1 \).

A3. Homogeneity. \( \phi(\lambda s_1, \ldots, \lambda s_I) = \lambda \phi(s_1, \ldots, s_I) \) is homogeneous of degree 0 in \( s = (s_1, \ldots, s_I) \).

A4. Monotonicity. If \( s_i \geq s_i, \) then \( \phi_i \geq \phi_k; \) and if \( s_i \geq s_k, \) then \( \phi(s_1, \ldots, s_i, \ldots, s_I) \geq \phi(s_1, \ldots, s_k, \ldots, s_I) \).

A5. Unanimity. There exists a \( \gamma \geq 1 \) s.t. \( \phi_i = 1 \) if \( s_i > \gamma \sum s_j \).

A6. Convexity 1. \( \phi(s_1, \ldots, s_I) \) is convex in \( s \) over the domain \( \{ (s_1, \ldots, s_N) \mid \phi(s_1, \ldots, s_N) < 1 \} \).

A7. Convexity 2. Consider a firm with 1 individual of size \( s_I \) and \( I \) individuals of size \( s_2 < s_1 \). Let \( s = s_1/s_2 \), and let \( \Phi(s, L) \) be benefits to an individual of size \( s \) in such a firm. Then \( \Phi \) is twice continuously differentiable, strictly convex in \( L \), and \( \left( \Phi_k - L \Phi_L \right) > 0 \) over the domain \( \{ (s, L) \mid \Phi(s, L) < 1 \} \).

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31. Without this assumption, all results would hold, but in a slightly less general form; propositions would have to be stated in terms of the sizes of both types of shareholders \( n \) and \( m \) instead of the ratio \( s = n/m \).

32. Subscripts represent partial derivatives. We take \( \Phi^i(s, L) \) to be defined on the domain \( \mathbb{R}_+^2 \). More precisely, \( \Phi^i(s, L) \) is defined on \( \mathbb{R}_+ \times \mathbb{Z}_+ \), and this assumption should be stated as: there exists an extension of \( \Phi^i(s, L) \) to \( \mathbb{R}_+^2 \) with the above properties.


