Topology optimization of continuum structures under buckling constraints

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Abstract

This paper presents a study on topology optimization of continuum structures under buckling constraints. New algorithms are developed for minimization of structural compliance considering constraints on volume and buckling load factors. The SIMP (Solid Isotropic Material with Penalization) material model is employed and nodal relative densities are used as topology design variables. A new approach based on the eigenvalue shift and pseudo mode identification is proposed for eliminating the effect of pseudo buckling modes. Two-phase optimization algorithms are also proposed for achieving better optimized designs. Numerical examples are presented to illustrate the effectiveness of the new methods.

1. Introduction

Structural strength, stiffness, and stability are three of the important factors considered for assessing the design of a structure. Naturally, it is important to consider structural stability in the optimization process. Recently, buckling optimization has drawn more research attention.

For trusses, frames, and other built-up structures consisting of bars and beams, much work has been done to consider the stability requirements in structural optimization, such as size optimization of trusses and frames [1], shape optimization of columns, truss or built-up structures [2–4] and topology optimization of truss structures [5–7].

Neves et al. [8] and Min and Kikuchi [9] have considered structural stability in topology optimization of continuum structures. They investigated the reinforcement of a structure to increase its overall stability. Neves et al. [10] extended their earlier work to the buckling optimization of periodic material micro-structures. Geometrically nonlinear models have also been introduced into the optimization of continuum structures against buckling [11–15]. In addition, optimization of composite structures was considered by Lindgaard and Lund [16,17].

A common problem in topology optimization using the SIMP material model is the appearance of pseudo buckling modes in low-density regions. Neves et al. [8] suggested ignoring the geometrical stiffness matrices of elements with densities and principal stresses smaller than predefined threshold values. Meanwhile, they indicated that the predefined values might have a significant influence on optimization results. Bendsøe and Sigmund [18] pointed out that doing this might cause solution oscillations due to abrupt changes of objective functions and sensitivities. In order to avoid the discontinuity caused by such a cut-off method, they suggested the use of different penalization schemes for element stiffness matrix and geometric stiffness matrix. Currently this method appears to be a standard solution for this problem and has been used by many researchers, e.g. Lindgaard and Lund [19]. However, Zhou [20] showed that it might be difficult to select an appropriate parameter value for the expression of penalization in calculating accurate buckling load factors.

Pseudo eigenmodes may also appear in the optimization of eigenfrequencies in vibration problems [21]. To eliminate these pseudo modes, some methods of modifying element stiffness matrix and/or mass matrix in low-density regions have been proposed and details of these methods can be found in the research literature, e.g. [21–23]. A topology optimization problem considering buckling differs from the one considering vibration modes and is more complex as element geometrical stiffness matrices are dependent on element stresses, which depend both on the structure itself and on the loading condition. In contrast, element mass matrices used in frequency analysis are dependent on material distribution only.

In this paper, the pseudo buckling mode problem is investigated and a new method combining eigenvalue shift and pseudo mode identification is proposed. An optimization formulation for
minimizing the structural compliance under material volume and buckling load factor constraints is used in the study.

This paper is organized as follows: In Section 2 the optimization formulation and material model used are presented. In Section 3, the finite element model employed for the structural analysis is introduced. In Section 4, expressions for the sensitivity of constraint functions and objective function are derived. In Section 5, some of the existing methods for dealing with pseudo buckling modes are briefly discussed and a new approach is proposed. In Section 6, new optimization algorithms are developed. In Section 7, two numerical examples are presented to demonstrate effectiveness of the proposed methods. Finally, concluding remarks are made.

2. Problem formulation and material interpolation scheme

2.1. Optimization problem formulation

The topology optimization of continuum structure may often generate designs with slender components when the allowed material volume fraction is small. If compressive stresses occur in these structural components, structural buckling may present serious safety concerns. Therefore, structural stability requirements should be considered in the optimization. The mathematical formulation of the compliance minimization problem of continuum structures with constraints on the material volume and buckling load factors can be stated as

\[
\begin{align*}
\text{find} & \quad \rho = (\rho_1, \rho_2, \cdots, \rho_N) \\
\text{min} & \quad C = \mathbf{F}^T \mathbf{U} = \mathbf{U}^T \mathbf{K} \mathbf{U} \\
\text{s.t.} & \quad \mathbf{K} \mathbf{U} = \mathbf{F} \\
& \quad \min |\alpha_j| \geq \frac{\lambda}{2} > 0 \\
& \quad V(\rho) \leq V_0 \\
& \quad 0 < \rho \leq \rho_i \leq 1 \quad i = 1, 2, \ldots, N
\end{align*}
\]

(1)

where \( \rho_i \) (\( i = 1, 2, \ldots, N \)) are design variables of relative material density; \( N \) is the number of design variables; \( C \) is the structural compliance; \( \mathbf{U} \) and \( \mathbf{F} \) are the global displacement and force vectors; \( \alpha_j \) is the \( j \)th buckling load factor corresponding to the given load cases; \( J \) is a set of indices of the buckling mode considered in the optimization; \( \lambda \) denotes the lower bound of buckling load factors; \( V(\rho) \) is the total material volume of the structure; \( V_0 \) is the upper bound of material volume; and \( \rho \) is the lower bound of design variables, e.g., \( \rho = 0.001 \).

Through the introduction of an explicit constraint condition on buckling load factors, designs that fail to satisfy stability requirements will be excluded from the feasible solution set. Theoretically, different levels of safety margins can be achieved by using different lower bound values. For example, if \( \frac{\lambda}{2} = 1 \), the optimized structure will be at a critical state under normal service conditions; if \( \frac{\lambda}{2} > 1 \), the structure will be stable under normal service conditions with a bigger safety margin for a bigger \( \frac{\lambda}{2} \); if \( 0 < \frac{\lambda}{2} < 1 \), the structure may buckler under normal service conditions, but cannot be a mechanism.

The buckling mode index set \( J \) is introduced for two reasons. Firstly, when an applied load always points in the same direction, negative loading factors are meaningless and in this case, set \( J \) should contain only the modes with positive load factors. Secondly, when pseudo modes are among the calculated buckling modes, the corresponding mode indices must be excluded from set \( J \) as these modes are not real and should be ignored.

2.2. Material interpolation

It is possible to obtain continuous material distributions by using nodal relative densities as topology design variables [24]. It is noted that Kang and Wang [25] have presented a more general density interpolation strategy for topology optimization using nodal design variables and Shepard interpolation. In this study, a more conventional interpolation scheme based on element nodal values and shape functions is used. Within the \( e \)th element, the relative density distribution is expressed as

\[
\rho^e(x, y) = \sum_{k=1}^{NN} N_k(x, y) \rho_k^e
\]

(2)

where \( \rho_k^e \) denotes nodal density value at the \( k \)th node of the element, \( NN \) is the number of nodes in the element, and \( N_k(x, y) \) is the element shape function for the \( k \)th node.

Using the SIMP material model, the elasticity matrix at point \( (x, y) \) is expressed in terms of material relative density \( \rho^e(x, y) \)

\[
E(x, y) = \rho^e(x, y)^p E_0
\]

(3)

where \( E_0 \) is the elasticity matrix of the isotropic solid elastic material, and \( p > 1 \) is a penalization exponent number.

3. Finite element analysis methods

In this section, the finite element model for structural analyses and the computation of buckling load factors using hybrid stress element is briefly introduced.

3.1. Finite element model

When the nodal design variable is employed, the checkerboard patterns can be avoided naturally. However, a “layering” or “islanding” phenomenon of black and white regions in the design domain may appear [24]. Deng et al. [26] showed that this problem could be effectively avoided by replacing the conventional four-node displacement-based quadrilateral element with a hybrid stress element. The same approach is taken in this study, and in this section, the basic theory and formulation of the hybrid stress element to be used will be summarized.

Pian and Sumihara [27] developed a four-node hybrid stress finite element for homogeneous plane problems. Independent element stress and displacement fields are defined and can be expressed as

\[
\sigma = \{\sigma_x, \sigma_y, \tau_{xy}\}^T = \Phi \beta
\]

(4)

\[
\mathbf{u} = \{u_x, u_y\}^T = \mathbf{N} \mathbf{d}
\]

(5)

where \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) are stress components, \( u_x \) and \( u_y \) are displacement components, \( \Phi \) and \( \mathbf{N} \) are interpolation matrices for element stress and displacement fields, respectively, \( \beta \) is an element stress parameter vector, and \( \mathbf{d} \) is the nodal displacement vector.

Based on the Hellinger–Reissner variational principle, the following expressions for element stiffness matrix \( \mathbf{K}_e \) and the stress parameter vector \( \beta \) can be derived

\[
\mathbf{K}_e = G_e \mathbf{H}_e^{-1} \mathbf{G}_e
\]

(6)

\[
\beta = H_e^{-1} \mathbf{G}_e \mathbf{d}
\]

(7)

where matrices \( \mathbf{G}_e \) and \( \mathbf{H}_e \) are defined as

\[
\mathbf{G}_e = \int_{-1}^{1} \int_{-1}^{1} \Phi^T \mathbf{B} \mathbf{J} \mathbf{d} \mathbf{z} \mathbf{d} \eta
\]

(8)

\[
\mathbf{H}_e = \int_{-1}^{1} \int_{-1}^{1} \Phi^T \mathbf{S}_0 \Phi \frac{1}{|\rho^e(\xi, \eta)|} \mathbf{E}_0 \mathbf{d} \mathbf{z} \mathbf{d} \eta
\]

(9)
where $J$ is Jacobian matrix, $|J|$ is the determinant of $J$, $B$ is the strain-displacement matrix, $S_0$ is the inverse of elasticity matrix $E_0$, i.e., $S_0 = E_0^{-1}$, and $t_0$ is the thickness of the structure.

### 3.2. Element continuum stiffness matrix

For a plane continuum structure, the element geometric matrix can be expressed as [28]

$$K_G = \int_{-1}^{1} \int_{-1}^{1} g^T \left[ \begin{array}{cc} S & 0 \\ 0 & S \end{array} \right] g \, d\eta \, d\xi \tag{10}$$

where submatrix $S$ and matrix $g$ are defined as

$$S = \left[ \begin{array}{cc} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{array} \right]$$

$$g = \Gamma \left[ M_1 \, M_2 \, M_3 \, M_4 \right]$$

in which $\Gamma$ and $M_i$ ($i = 1, 2, 3, 4$) are matrices defined below

$$\Gamma = \left[ \begin{array}{c} J^T \\ 0 \\ 0 \end{array} \right]$$

$$M_i = \left[ \begin{array}{ccc} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial x} \frac{\partial N_i}{\partial y} \end{array} \right]^T$$

From Eqs. (4) and (7), the stress vector can be calculated as follows

$$\sigma_i = fH_{ie}^{-1} \sigma_d$$

Now, with a mapping function $\Theta : R^3 \rightarrow R^{4 \times 4}$ defined as

$$\Theta = \left[ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right]$$

the element geometric matrix in Eq. (10) can be rewritten as

$$K_G = \int_{-1}^{1} \int_{-1}^{1} g^T \Theta(\sigma_j) g \, d\eta \, d\xi \tag{17}$$

### 3.3. Determination of buckling load factors

The linear buckling load factors can be calculated from the following equation

$$(K + \lambda_J K_c) \psi_j = 0 \tag{18}$$

where $K$ is the global stiffness matrix of the structure, $K_c$ is the global geometric stiffness matrix, and $\lambda_j$ and $\psi_j$ are the $j$th buckling load factor and the corresponding buckling mode vector.

Normally the buckling modes are ordered according to the magnitudes of buckling load factors and $\lambda_1$ will be the smallest.

### 4. Sensitivity analysis

In this section, the sensitivities of compliance, buckling load factors and volume with respect to nodal design variables are derived.

#### 4.1. Sensitivity of compliance

When the applied loads are independent of design variables, the sensitivity of structural compliance can be expressed as

$$\frac{\partial C}{\partial \rho_i} = -U^T \frac{\partial \mathbf{K}}{\partial \rho_i} U = -\sum_{e=1}^{N_e} \left( \mathbf{u}_e \right)^T \frac{\partial \mathbf{K}_e}{\partial \rho_i} \mathbf{u}_e \tag{19}$$

where $\mathbf{K}_e$ and $\mathbf{u}_e$ are the stiffness matrix and the displacement vector of element $e$, respectively, $N_e$ is the number of elements used to discretize the design domain.

From Eq. (6), we get

$$\frac{\partial \mathbf{K}}{\partial \rho_i} = G_i \frac{\partial \mathbf{H}_e^{-1}}{\partial \rho_i} G_e$$

Then differentiating the identical equation $H_e^{-1} = I$ with respect to $\rho_i$, we can get

$$\frac{\partial \mathbf{H}_e^{-1}}{\partial \rho_i} = -H_e^{-1} \frac{\partial H_e}{\partial \rho_i} H_e^{-1}$$

From Eq. (9), we can get the following expression

$$\frac{\partial H_e}{\partial \rho_i} = -\int_{-1}^{1} \int_{-1}^{1} \Phi^T \frac{\partial \mathbf{S}_e}{\partial \rho_i} \Phi \frac{\partial \mathbf{J}}{\partial \rho_i} \mathbf{d} \xi \, d\eta$$

where

$$\frac{\partial \mathbf{d} \xi}{\partial \rho_i} = \begin{cases} 0 & \text{if } i \text{ is not adjacent to element } e \\ \frac{N_e(\xi, \eta)}{i} & \text{if } i \text{ is the } k \text{th node of element } e \end{cases}$$

Then using Eqs. (19)–(22), we can obtain the sensitivity of the compliance with respect to the nodal design variables.

#### 4.2. Sensitivity of buckling load factors

Introducing an auxiliary variable $\kappa_j = -1/\lambda_j$ [29], Eq. (18) can be rewritten as

$$(\mathbf{K}_c(\rho, U) - \kappa_j K_c(\rho)) \psi_j = 0 \tag{24}$$

Note that for the current optimization problem, both of the two global stiffness matrices are functions of the current design $\rho$ and the global geometric stiffness matrix depends also on $U$, the nodal displacement vector for the given loading case.

If the eigenvalue is unimodal, the sensitivity of auxiliary variable $\kappa_j$ with respect to variable $\rho_i$ can be expressed as [29]

$$\frac{\partial \kappa_j}{\partial \rho_i} = \frac{\psi^T_j \left( \frac{\partial \mathbf{K}}{\partial \rho_i} - \frac{\partial \mathbf{K}_c}{\partial \rho_i} \right) \psi_j - \psi^T_j \frac{\partial \mathbf{K}}{\partial \rho_i} \psi_j}{\psi^T_j \frac{\partial \mathbf{K}_c}{\partial \rho_i} \psi_j} \tag{25}$$

where $\psi_j$ is the eigenvector, and $\psi^T_j \frac{\partial \mathbf{K}_c}{\partial \rho_i} \psi_j$ is the adjoint displacement vector. Adjoint displacement vectors are determined by solving the following equation

$$\mathbf{K}_c \psi_j = \mathbf{P}_j \tag{26}$$

where

$$\mathbf{P}_j = \psi^T_j \frac{\partial \mathbf{K}_c}{\partial \rho_i} \psi_j$$

in which $d$ is the number of degrees of freedom of the structure. Note that the eigenvectors must satisfy the orthonormalization condition, i.e., $\psi^T_j \mathbf{K}_c \psi_k = \delta_{jk}$, $\delta_{jk}$ is Kronecker’s delta.

By differentiating Eq. (17), we get

$$\frac{\partial K_G}{\partial \rho_i} = \int_{-1}^{1} \int_{-1}^{1} g^T \frac{\partial \Theta(\sigma)}{\partial \rho_i} g \, d\eta \, d\xi \tag{27}$$

Since mapping $\Theta$ is linear, the derivatives of $\Theta(\sigma)$ with respect to $\rho_i$ can be expressed as
\[ \frac{\partial \Theta(\sigma_\ell)}{\partial p_i} = \frac{\partial \Theta(\Phi H_e^{-1} G_{\ell u})}{\partial p_i} = \Theta(\Phi \frac{\partial H_e^{-1} G_{\ell u}}{\partial p_i}) \]  

(29)

The sensitivity of the global geometrical stiffness matrix can then be given as

\[ \frac{\partial K_e}{\partial p_i} = \sum_{\ell=1}^{N_e} \frac{\partial K_{e\ell}}{\partial p_i} = \sum_{\ell=1}^{N_e} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \Theta(\sigma_\ell)}{\partial p_i} g_J |d| \, d\eta 
\]

\[ = \sum_{\ell=1}^{N_e} \int_{-1}^{1} \int_{-1}^{1} g^\top \Theta(\Phi \frac{\partial H_e^{-1} G_{\ell u}}{\partial p_i}) g_J |d| \, d\eta \]  

(30)

Using Eqs. (21) and (30), we can calculate \( \frac{\partial K_e}{\partial p_i} \).

Following the same procedure, the sensitivity of the elemental geometrical stiffness matrix with respect to displacement component \( u_j (j = 1, 2, \ldots, d) \) can be calculated

\[ \frac{\partial K_{e\ell}}{\partial u_j} = \int_{-1}^{1} \int_{-1}^{1} g^\top \Theta(\Phi H_e^{-1} G_{\ell u}) g_J |d| \, d\eta \]

\[ = \int_{-1}^{1} \int_{-1}^{1} g^\top \Theta(\Phi H_e^{-1} G_{\ell u}) g_J |d| \, d\eta \]  

(31)

where \( I \) is an \( 8 \times 1 \) vector. If \( u_j \) is not a nodal displacement of the element, then \( I \) is a zero vector. Otherwise \( I \) is a unit vector and the \( k \)th component is one if the \( u_j \) is the \( k \)th nodal displacement component of the element.

Once the sensitivities of the introduced auxiliary variables \( \kappa_j = -1/\lambda_j \) are obtained, we can use the chain rule to calculate the sensitivities of eigenvalues \( \lambda_j \) as follows

\[ \frac{\partial \lambda_j}{\partial p_i} = \frac{\partial \lambda_j}{\partial \kappa_j} \frac{\partial \kappa_j}{\partial p_i} = \lambda_j^2 \frac{\partial \kappa_j}{\partial p_i} \]  

(32)

If the eigenvalue is multimodal with multiplicity \( m \) greater than one, individual eigenvalues may no longer be differentiable functions of the design variables. In such situations, the method proposed by Gravesen et al. [30] can be used. If \( \kappa_1 = \kappa_2 = \ldots \), we can calculate sensitivities of functions \( \kappa_1 + \kappa_2 \) and \( \kappa_1 \kappa_2 \) using the following expressions

\[ \frac{\partial (\kappa_1 + \kappa_2)}{\partial p_i} = \psi_1 \left( \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} - \frac{1}{\lambda_1 \lambda_2} \frac{\partial \lambda_1}{\partial p_i} \frac{\partial \lambda_2}{\partial p_i} \right) \psi_1 - \psi_1^2 \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} \]

\[ + \psi_2 \left( \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} - \frac{1}{\lambda_1 \lambda_2} \frac{\partial \lambda_1}{\partial p_i} \frac{\partial \lambda_2}{\partial p_i} \right) \psi_2 - \psi_2^2 \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} \]  

(33)

\[ \frac{\partial (\kappa_1 \kappa_2)}{\partial p_i} = \kappa_2 \left( \psi_1 \left( \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} - \frac{1}{\lambda_1 \lambda_2} \frac{\partial \lambda_1}{\partial p_i} \frac{\partial \lambda_2}{\partial p_i} \right) \psi_1 - \psi_1^2 \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} \right) \]

\[ + \kappa_1 \left( \psi_2 \left( \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} - \frac{1}{\lambda_1 \lambda_2} \frac{\partial \lambda_1}{\partial p_i} \frac{\partial \lambda_2}{\partial p_i} \right) \psi_2 - \psi_2^2 \frac{\partial \kappa_1}{\partial p_i} \frac{\partial \kappa_2}{\partial p_i} \right) \]  

(34)

Other methods for dealing with multiple eigenvalues can be found in the research literature [8,10,23,31].

4.3. Sensitivity of total material volume

The total volume of material used in the structure can be calculated as

\[ V = \sum_{e=1}^{N_e} \int_{-1}^{1} \int_{-1}^{1} \rho_v |\eta| |d| \, d\eta \]  

(35)

Its sensitivity can then be expressed as

\[ \frac{\partial V}{\partial p_i} = \sum_{e=1}^{N_e} \frac{\partial V_e}{\partial p_i} = \sum_{e=1}^{N_e} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \rho_v |\eta| |d|}{\partial p_i} |d| \, d\eta \]  

(36)

\[ \frac{\partial \rho_v |\eta| |d|}{\partial p_i} = \sum_{e=1}^{N_e} \frac{\partial V_e}{\partial p_i} = \sum_{e=1}^{N_e} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \rho_v |\eta| |d|}{\partial p_i} |d| \, d\eta \]

where \( \frac{\partial \rho_v |\eta| |d|}{\partial p_i} \) is given in Eq. (23).

5. Methods for dealing with pseudo buckling modes

Pseudo buckling modes usually appear in low-density regions during the optimization process. In this section, some existing methods for dealing with this problem are first investigated through a simple example, and then a new approach is proposed.

5.1. A discussion on existing methods for dealing with pseudo buckling modes

For an investigation of existing methods for dealing with local pseudo buckling modes, the 2D portal frame structure shown in Fig. 1 is considered. Its first buckling load factor is to be calculated. The frame with a thickness of 10 mm is clamped at the bottom, and is modeled with a regular 40 × 20 mesh of four-node plane stress elements. A downward concentrated force \( F \) is applied at the center on the top edge. The outer frame with a thickness of 5 mm has fully dense material \( \rho = 1.0 \) and the inner region has a density \( \rho = 0.001 \).

For a real structural design, the low-density region should be ignored and only the buckling modes occurring in the solid parts or the regions with a high relative density value should be considered. In order to distinguish these modes from the pseudo modes, we call them “real” buckling modes, and their corresponding eigenvalues “real” buckling load factors. Thus, the first real buckling load factor is calculated by using a model with elements of the low density excluded, and a buckling load factor of 2.41 is obtained. This is taken as a reference solution.

For most of the existing methods for suppressing pseudo modes, different schemes are used for choosing modulus for stiffness matrix \( E_v^e \) and geometric stiffness matrix \( E_v^g \). In this study, the three schemes in Table 1 are considered.

The results are summarized in Table 2. Quite different results have been obtained. Even with the same method, we may still obtain very different results when different parameter values are used.

Method (a) uses a material model without penalization on intermediate density. For this problem, it gives a result 156.3% higher than the reference solution. The significantly higher buckling load factor is due to the contribution of the inner region, which may act as a thin membrane. As the material modulus for the low density region is not scaled down with a small factor, say using penalization exponent number \( p > 1 \), pseudo buckling modes can be avoided. However, this simple treatment may lead to optimized designs with many gray regions because no penalization is applied. The existence of these gray regions may lead to the over-estimation of buckling load factors, as shown in this example (see Zhou [20]). Therefore, much care should be taken when using this method.

\[ E = 2.0 \text{GPa} \]
\[ \nu = 0.3 \]
\[ W = 200 \text{mm} \]
\[ H = 100 \text{mm} \]
\[ t_c = 10 \text{mm} \]
\[ F = 100 \text{N} \]

Fig. 1. Portal frame.
Although method (b) produces a similar result as the reference solution, it may cause solution oscillations in an optimization process [18]. Besides parameter $\rho_1$, this method requires an additional parameter of critical normalized stress value $\sigma_c$. Geometrical stiffness matrices of low-density elements with normal stress less than $\sigma_c$ will be ignored in the optimization process [8,10]. Generally speaking, the critical normalized stress $\sigma_c$ is difficult to predefine, and an improper value may result in different topology results [8]. Method (c) can avoid the discontinuity problem of method (b). However, as the introduced parameter $\rho_2$ plays a critical role in avoiding the appearance of pseudo modes, using an improper value may lead to erroneous results [19,20]. A large value of $\rho_1$ may result in unrealistically high stiffness of void elements as shown in this example (e.g. $\rho_1 \gg 10^{-5}$). On the other hand, a small value of $\rho_1$, such as $\rho_1 \leq 10^{-8}$ for this example, is insufficient for avoiding the pseudo buckling mode. Hence, the difficulty of this method is to choose a proper value $\rho_1$ to calculate accurate real buckling load factors. Although the lower bound of relative density (say, 0.001) is often used for $\rho_1$, such a value may still be too large and as a result, the structural stability can be overestimated.

On the other hand, if buckling load factors of the portal frame are calculated by using the standard SIMP model with $p = 3$ and including all the elements in the low-density region, the first 48 buckling modes obtained are pseudo modes with very small eigenvalues and the 149th mode with a load factor of 2.41 is the first real mode. Therefore, it is important to find reliable methods for dealing with these pseudo modes.

As similar situations may occur for other topology optimization problems, the development of effective methods for dealing with pseudo modes is an important research topic. A new approach is proposed in the following sections.

5.2. Methods for eliminating the effects of pseudo modes

An investigation into the occurrence and characteristics of pseudo buckling modes reveals that pseudo modes have the following two important features:

1. Some pseudo buckling modes have eigenvalues close to zero, or more accurately, much smaller than those of the real modes. As there may be many such pseudo modes, a large number of eigenvalues have to be determined to include some of the real modes.
2. The deformation mainly occurs in low-density regions, and as a result, the modal strain energy in low-density regions makes a major contribution to the total modal strain energy.

Based on these two observations, a new strategy is proposed.

5.2.1. Calculation of candidate buckling modes

In linear buckling analysis, it is a common practice to calculate only the first several eigenmodes to reduce the computational cost. If a model has many pseudo buckling modes, it will be necessary to calculate a large number of eigenvalues. Otherwise, it is possible that all those calculated are pseudo modes, which are useless. Thus, the existence of pseudo modes may cause a dramatic cost increase in eigenvalue calculation.

However, if we know the range of real buckling load factors, this issue could be easily resolved by applying the eigenvalue-shift technique [32]. By applying an appropriate shift value, real low-order buckling modes can be calculated at a reasonable cost as pseudo models with very small eigenvalues are excluded from the calculation.

When a uniform material distribution is used as the initial design, no pseudo buckling modes will appear at the early stage of the optimization, because there are no low-density regions. In this case, the range of real buckling load factors is easily available. When a non-uniform initial design is used, the range of real eigenvalues can still be determined, for example, by using the method proposed by Neves et al. [8]. One can also calculate a relatively large number of buckling load factors in the first iteration and find real low-order buckling modes using the identification technique presented in Section 5.2.2. Therefore, it is always possible to determine the real low-order buckling modes and calculate the corresponding load factors for the initial design.

Let $\lambda^{(k-1)}$ be the first real buckling load factor for the last design. Assuming that the design will not change dramatically, we can use $\lambda^{(k-1)}$ as the shift to calculate $M$ buckling modes of a new, modified design, and then pick the smallest eigenvalue of the real modes. Obviously, it is more likely that we can find the real first buckling mode if a larger $M$ is used. However, considering that it requires more computer resources and time to compute more eigenvalues than necessary, a smaller number is preferred. Based on the assumption that optimized designs change only moderately between iterations, it is possible to choose a reasonably small number for $M$. In this study, $M = 50$ is used for all examples, which means that only 50 eigenmodes are determined in the eigenvalue extraction.

Therefore, by applying an eigenvalue shift, possible low-order pseudo modes can be excluded from the eigenvalue extraction computation.

5.2.2. Pseudo buckling modes identification

Aside from low-order pseudo models with very small eigenvalues, other pseudo modes among the $M$ eigenmodes may still exist. These high-order pseudo modes must be identified to avoid a situation where a wrong constraint is introduced. We can make use of the second feature of pseudo buckling modes to determine whether a mode is real or not: As low-density elements (regions) make the major contribution to the total modal strain energy of a pseudo mode, we can divide the modal strain energy into two parts, one from the low-density regions and the other.
from the rest of the structure. We can then identify this mode based on the energy contributions of these two parts.

First, a threshold parameter $\rho_i$ is introduced, and all of the nodes can be divided into two sets based on nodal relative density values, i.e., nodes with low densities $N_l = \{i | \rho_i \leq \rho_1, 1 \leq i \leq N\}$ and nodes with high densities $N_h = \{i | \rho_i > \rho_1, 1 \leq i \leq N\}$.

Then, the degrees of freedom can be divided into two groups based on the grouping of nodes [33], and a buckling mode vector can be decomposed into two vectors:

$$\Psi_j = \Psi_{j0} + \Psi_{jh}$$

where $\Psi_{j0}$ is composed of displacement components for degrees of freedom of nodes in set $N_l$ and zero elements, and $\Psi_{jh}$ is composed of the displacement components for degrees of freedom of nodes in set $N_h$ and zero elements.

Hence, the modal strain energy ratio of low-density regions $r_j^l$, defined as the ratio of contribution from the nodes in $N_l$ to the total modal strain energy, is given by

$$r_j^l = \frac{W_j^l \Psi_{j0}^T K \Psi_{j0} + W_j^h \Psi_{jh}^T K \Psi_{jh}}{W_j^l \Psi_{j0}^T K \Psi_{j0}}$$

The pseudo buckling mode identification criterion can be stated as

$$r_j^l > MW_l$$

where $MW_l$ is a predefined parameter with a value between 0 and 1. If the modal strain energy ratio of low-density regions is bigger than the value, the corresponding mode is regarded as pseudo mode; otherwise, the mode is treated as real.

A region with a relative density less than $\rho_1$ is usually regarded being of low-density, and all numerical tests conducted in this study have shown that the threshold value $\rho_1 = 0.1$ is appropriate. For a clear black-white design in which the design variables are equal to either one or the lower bound $\rho$, the modal strain energy ratio for a pseudo mode can be bigger than 0.98, as demonstrated by the example in Section 5.1. During the optimization iteration, it is very likely that a topology design contains gray regions. For a pseudo mode, deformation in these regions may be small but is not zero, causing a reduction in the modal strain energy ratio that may have an effect on a pseudo mode check. In order to enhance the reliability of the mode identification results, it is necessary to decrease the value of $MW_l$. In this study, the selected parameter values $\rho_1 = 0.1$ and $MW_l \in [0.6, 0.7]$ are used and prove to be appropriate for all of the conducted numerical tests.

The proposed identification method is used on the 150 calculated buckling modes of the example structure in Section 5.1, and all the modes are correctly identified. The pseudo mode identification is an important component of the optimization algorithm proposed in the next section, and numerical experiments show that the identification method is very effective and reliable.

6. Optimization algorithm

Based on the finite element analysis and sensitivity analysis, topology optimization problem (see Eq. (1)) can now be solved by using a gradient-based optimization algorithm. Optimization algorithms based on the well-known MMA optimization solver [34] are developed. Different optimization strategies are also proposed for achieving high-quality local solutions.

6.1. Computational procedure

Compared with the conventional problem of compliance minimization under material volume constraint, the optimization model in Eq. (1) has an additional buckling constraint on the smallest buckling load factor. It is both simple and natural to just add this new constraint equation and then use an existing optimization algorithm to solve the new problem. Based on this idea, the iterative solution procedure in Fig. 2 is proposed for the solution of the topology optimization problem in Eq. (1). To start with, the design domains are discretized and the optimization parameters are defined. In the FE analysis of the initial design in Step 2 and the current design in Step 5, linear static responses, the ‘real’ buckling load factors $\lambda_j$ and corresponding mode vectors $\Psi_j$, the structural compliance $C$ and the volume of material used in the design $V$ are all determined. In Step 3, the sensitivities of the object

![Fig. 2. Flow chart of the iterative solution procedure.](image)

![Fig. 3. 2-D continuum structure under distributed load (a) problem description and (b) topology from compliance minimization subjected to volume constraints.](image)
and constraint functions with respect to the nodal design variables are computed. In addition, we will check the multiplicity of the first eigenvalue. If the multiplicity $m$ is bigger than one, we will use multimodality formulation to calculate its sensitivities. In Step 4, the design is modified by using optimization solver MMA.

To check the solution convergence, the following criteria are used:

(a) the maximum change in the design variables between two consecutive iterations is smaller than a predefined tolerance $\varepsilon_p$, i.e., $\|\rho^{k+1} - \rho^k\|_\infty \leq \varepsilon_p$;

(b) the change in the objective function between two consecutive iterations is smaller than a predefined tolerance $\varepsilon_c$, i.e., $|C^{k+1} - C^k|/C^k \leq \varepsilon_c$;

(c) all of the constraint conditions are satisfied.

6.2. Improved optimization strategies

While the algorithm presented earlier is applicable to the buckling optimization problem, as will be shown through numerical examples in the next section, it does not always work well. As there is usually a confliction between the requirements for structural stiffness and stability, the buckling constraint should not be treated just like the one in the material volume, implying that refined algorithms are required to achieve more optimized designs. It is proposed that the optimization process is separated into two phases and at each phase, different optimization models and/or material models are employed. In the present paper, two improved algorithms are given and compared with the simple one phase algorithm shown in Fig. 2. The initial value for nodal design variables in all of the three algorithms is uniform with the volume fraction. The three algorithms investigated are as follows:

(a) Algorithm A: solve optimization problem (Eq. (1)) by following the procedure shown in Fig. 2 with no changes.

(b) Algorithm B: separate the optimization process into two phases and use different optimization models. In the first phase, solve the conventional problem of compliance minimization under the volume constraint by ignoring the buckling constraints. In the second phase, solve the original problem (Eq. (1)) with the solution from the first phase as the initial design.

![Fig. 4. Compliance vs the lower bound of buckling load factor.](image)

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Optimized topologies for different buckling constraints.</th>
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<tbody>
<tr>
<td>Algorithm</td>
<td>$\lambda = 0.6$</td>
</tr>
<tr>
<td>A</td>
<td><img src="image1" alt="Image" /></td>
</tr>
<tr>
<td>B</td>
<td><img src="image7" alt="Image" /></td>
</tr>
<tr>
<td>C</td>
<td><img src="image13" alt="Image" /></td>
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</tbody>
</table>
(c) Algorithm C: separate the optimization process into two phases and use different material penalization for calculating buckling load factors. In the first phase, use normal material penalization for calculating compliance and its sensitivity, but do not penalize the material modulus for calculating the buckling load factor (i.e. use $p = 1$). In the second phase, use the normal material penalization to solve the original problem (Eq. (1)) with the solution from the first phase as the initial design.

It is noteworthy that, for both algorithms B and C, a 'pre-solve' phase is added and the second phase is the same as the solution of the original problem by using algorithm A, but with a different initial design from the first phase. In the first phase, the buckling constraints are either ignored as in algorithm B or considered but in a modified form as in algorithm C. In contrast, the initial design for algorithm A is a uniform distribution in the design domain.

The numerical examples in the next section will show that by separating the optimization procedure into two phases, the solution may converge in fewer iterations and better performance of the optimized structures can be achieved.

7. Numerical examples

Two examples are considered in this section and the designs obtained by using the three algorithms are compared.

7.1. Example 1

The first example is the optimization of a 2D continuum column-like structure under distributed loads. The load and support conditions are shown in Fig. 3(a). The design domain is a rectangular area of unit thickness with height $H = 40$ and width $W = 20$. A distributed load $q = 0.05$ is applied at the top edge with a width $d = 2/3$. The design domain is discretized into $60 \times 120$ equally-sized square four-node elements. The material constants used are Young’s modulus $E = 1.0$ and Poisson’s ratio $\nu = 0.3$. The prescribed material volume fraction number is set to 0.35.

The initial value of all nodal design variables are set to the volume fraction. The optimized design of a conventional minimum compliance problem is shown in Fig. 3(b) and its corresponding first buckling load factor is 0.45. When considering buckling constraints, the parameters for convergence criterion are $\lambda_0 = 0.01$ and $\lambda_m = 0.001$ and the move limit on design variables is $m_p = 0.003$.

This optimization problem has been solved using the three algorithms for different lower bounds of buckling load factors. The curves of compliance versus the lower bound of buckling load factor are presented in Fig. 4, while the obtained topologies are shown in Table 3. It can be seen that the compliance increases as the lower bound of buckling load factor increases. This means that with a fixed amount of material, the improvement of structural stability can be achieved only by a reduction in structural stiffness. A comparison of the designs obtained with buckling constraints and those obtained with the volume constraint only reveals that the requirement on the structural stability tends to distribute the available material over a larger area. In contrast, the maximization of structural stiffness causes the material to distribute along the load transfer path.

From Fig. 4, it is found that the compliance values for the three solutions are very close for all of the considered lower bound values. However, the topologies obtained have some minor differences. This indicates that very likely, the solutions are just local optima, and for this particular problem, different local solutions have similar topologies and compliance values.

The iteration history curves in Fig. 5 show that algorithm A requires considerably more iterations than algorithms B and C for convergence. From the topological changes shown in Fig. 6, it can be found that with algorithm A, modulus penalization makes the buckling constraint harder to be satisfied and causes the material to distribute over a much larger region than required at the early stage of the optimization. At the later stage, the optimization algorithm will guide the design to change and produce better designs. However, the intermediate designs could be so different from the final design that the solution requires a large number of iterations to converge. In Fig. 5(b), we can see that in iterations 167–480, the objective function increases first and then decreases until the solution terminates. This is because different constraints are considered in the two phases of algorithm B. In the first phase, the buckling constraint is not considered at all, thus the optimized topology structure has high stiffness but the buckling constraints may not be satisfied. In the second phase, the optimization algorithm will steer the design to improve structural stability and may cause a reduction in stiffness. However, once the buckling constraints are satisfied, compliance will begin to decrease as optimization proceeds. Fig. 5(c) shows a drop in first buckling load factor at iteration 300. This sudden change is due to the switch of the solution phase. From this iteration onwards, the normal material penalization will be employed for buckling analysis. It can be seen from Fig. 6 that for algorithm B and C, at the first stage, the compliance of topological designs is relatively small and material is
mainly distributed along the load transfer path with a relatively low stability. At the later stage, some material is moved away from the load transfer path to improve structural stability.

The first three buckling modes of optimized structures for $k = 1.0$ are shown in Table 4. The first real buckling load factors for the three algorithms are all bimodal. This validates that the multimode sensitivity analysis methods presented at the end of Section 4.2 is effective.

### Table 4

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
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<tr>
<td>B</td>
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<td></td>
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<tr>
<td>C</td>
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The first three buckling modes of optimized structures for $\lambda = 1.0$ are shown in Table 4. The first real buckling load factors for the three algorithms are all bimodal. This validates that the multimode sensitivity analysis methods presented at the end of Section 4.2 is effective.

#### 7.2. Example 2

The short cantilever beam shown in Fig. 7(a) is considered. The design domain is a rectangular area of unit thickness with height $H = 2$ and width $W = 1$. A concentrated load $F = 0.005$ is applied to the center of the right edge. The design domain is discretized into $40 \times 80$ equally-sized square four-node elements. The material constants used are Young's modulus $E = 1.0$ and Poisson's ratio $v = 0.3$. The prescribed material volume fraction is set to 0.15.

The initial value of all nodal design variables are set to the volume fraction. The optimized design of a conventional minimum compliance problem is shown in Fig. 7(b) and the corresponding first buckling load factor is equal to 0.22.

The optimized designs obtained by employing three algorithms for different lower bounds of buckling load factors are shown in Table 5. It can be clearly seen that as the lower bound of buckling load factor increases, the structural member in compression becomes shorter and wider, resulting in stability improvement. At the same time, some gray regions appear. The appearance of gray region means that it is impossible to obtain a clear black-white design to satisfy the constraint.
one-phase algorithm and algorithm C is the best of the three. In addition, this example again validates the effectiveness of the new approach combining pseudo buckling mode identification and eigenvalue shift in dealing with pseudo buckling modes.

8. Conclusions

The compliance minimization problem under volume and stability constraints is considered. A new approach has been proposed for dealing with the well-known pseudo buckling mode problem. In addition, in consideration of the non-linear and non-convex nature of the problem, two-phase algorithms are suggested for achieving better local optimization solutions. Numerical examples are presented to show the effectiveness of the new algorithms.

It should be pointed out that the proposed approach for dealing with the pseudo buckling mode problem can be applied to vibration optimization problems [35]. Further, the two-phase optimization strategies are effective for improving the performance of optimized designs. The same idea could be useful for other engineering optimization problems, which may also be highly non-linear and non-convex.

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