Thermal shock resistance of functionally graded materials

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Abstract

Transient temperature field and associated thermal stresses in functionally graded materials (FGMs) are determined by a finite element/finite difference (FE/FD) method. Temperature-dependent material properties are taken into consideration. Explicit expressions for one-dimensional transient thermal conduction in some common elements, such as plate, shell and sphere, are given. These expressions are useful for material engineers and scientists to determine the thermal stresses and strength distributions in FGMs for high temperature applications. Thermal shock fracture of a FGM plate is analyzed when the plate is suddenly exposed to an environmental medium of a different temperature. The admissible temperature jump that the materials can sustain is studied using stress-based and fracture-toughness-based failure criteria. The critical parameters governing the level of the transient thermal stress in the medium are identified. The thermal shock resistance of the FGMs is analyzed using both maximum local tensile stress and maximum stress intensity factor criteria.

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1. Introduction

A functionally graded material (FGM) is usually a combination of two material phases that has a gradual transition from one material at one surface to another material at the opposite surface. This transition allows the creation of multiple properties (or functions) without any mechanically weak junction or interface. Furthermore, the gradual change of properties can be tailored to different applications and service environments. It is possible with these materials to obtain a combination of properties that cannot be achieved in conventional monolithic materials. This makes FGMs preferable in many applications.

FGMs are expected to be used for high temperature environments. Thermal shock conditions which arise during sudden heating or cooling of a solid can result in very high stresses. If the thermal transient is severe enough, fracture may occur. The degree of damage and strength degradation of materials subjected to severe fluctuating thermal environments is a major limiting factor in relation to service requirements and lifetime performance. Therefore, it is important to analyze the internal thermal stresses in FGMs and to evaluate their resistance to thermal loading such as thermal shock. The purpose is to control the thermal stress level for a given material system such that it does not suffer damage under a given thermal environment. In the past few years, FGMs under thermal environments have been extensively investigated. The first approach is based on the thermoelastic theory [1–5]. The severity of thermal shock in a medium is qualified by drawing a correlation between the magnitude of the maximum tensile stress in the medium and the likelihood of failure. This approach assumes that there are no pre-existing flaws inside the medium. Material properties are selected to avoid the initiation of fracture by thermal stresses. In general this requires

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materials with high values of tensile strength, thermal conductivity, and thermal diffusivity combined with low values of the thermal expansion coefficient, Young’s modulus and Poisson’s ratio. Material non-homogeneity can be chosen such that the thermal stress can be minimized.

The second approach to the determination of thermal stress resistance of the FGMs is concerned with the extent of crack growth. Many theoretical models of the crack problem under thermal stress conditions have been developed for FGMs [6–9]. This mainly involves the solution of two distinct boundary-value problems. The first is a standard potential problem in the classical theory of thermal conduction, and the second is the determination of the stresses caused by the uneven and/or unsteady heating of the various points within the solid. This is a more sophisticated fracture analysis and incorporates cracks into the transient stress analysis and thus the degree of severity of any giving thermal shock is characterized in terms of the stress intensity factor (SIF) which in turn is a function of the crack size and pertinent heat transfer and thermoelastic coefficients [10–12].

Due to mathematical difficulties, transient thermal effects on fracture of FGMs have not received much attention. It is generally accepted that the strength of a material under a transient thermal stress state depends on a number of material properties including thermal expansion coefficient, thermal conductivity, thermal diffusivity, fracture toughness and tensile strength, elastic properties, etc. For FGMs, the material non-homogeneity parameter, such as the gradient index, can also affect transient thermal stress and thermal fracture significantly. This paper presents an approach to the calculation of the thermal stresses and the extent of crack propagation in FGMs. The solutions for the theory include the conditions for thermal stress fracture initiation and continued crack propagation. A finite element method together with the finite difference technique is used to solve the system of time dependent equations. Solutions are thus obtained for the maximum local tensile stress equals the local tensile fracture toughness of the material.

2. Numerical solution to the transient temperature field and associated thermal stresses

A key feature of the mechanics that distinguishes FGMs from homogeneous materials is that the properties of the former vary spatially. Thus, thermo-mechanics analysis of FGMs is considerably more complex than corresponding homogeneous materials of the same specimen subjected to the same loading conditions. Therefore, numerical method for solving the temperature distribution is of great importance. In what follows, we introduce a finite element/finite difference (FE/FD) method to obtain the solution for the time-dependent temperature field.

2.1. Thermal conductivity equation

Suppose in a coordinate system $x_i$ ($i = 1, 2, 3$) there is a solid occupying a space $\Omega$ which is surrounded by a surface $S$. The temperature inside the solid may vary from point to point, and from time to time. Let $T(x_i, t)$ denote this temperature which is assumed to be a continuous function of the coordinates $x_i$ and time $t$. In the following, the summations over the indices $i$ and $j$ will be assumed when appearing twice in an equation. From the basic law of heat conduction $q_i = -k_{ij} \frac{\partial T}{\partial x_j}$ and the well-known equation for the heat flux $-q_i = Q + \rho c \frac{\partial T}{\partial t}$, we can obtain the equation controlling the temperature field:

$$k_{ij} \frac{\partial T}{\partial x_j} = Q + \rho c \frac{\partial T}{\partial t}.$$  

(1)

where $q_i$ are components of the heat flux vector $q$, $Q$ the internal heat generation rate per unit volume, $\frac{\partial T}{\partial x_j}$ the temperature gradients, $\rho(x)$ the mass density, $c(x)$ the specific heat, and the components of the thermal conductivity tensor $k(x)$ are denoted by $k_{ij}$ which are generally considered to be symmetric, i.e., $k_{ij} = k_{ji}$. If the solid is anisotropic, heat will not necessarily flow in the direction of the temperature gradient. For isotropic solids, $k_{ij}$ vanishes if $i \neq j$. The material properties $\rho$, $c$ and $k_{ij}$ are functions of space coordinates $x_i$.

The thermal conduction equation (1) must be solved for prescribed boundary and initial conditions. The initial condition specifies the temperature distribution at time zero, this is $T(x_i, 0) = T_0(x_i, 0)$. Heat conduction boundary conditions take several forms. The frequently encountered conditions are specified surface temperature and specified surface heat flow:

$$q_n = \bar{h}, \quad \text{on boundary} \ S_q, \quad (2a)$$

$$T = \bar{T}, \quad \text{on boundary} \ S_T, \quad (2b)$$

where $S_q + S_T = S$, and the over bar represents the known value, $n_i$ are components of unit vector $n$ normal to the exterior of $S$. Eq. (2a) indicates that on the boundary $S_q$, the thermal flux is prescribed ($\bar{h}$ is positive if it is directed towards the exterior of the body). Eq. (2b) indicates that the temperature on $S_T$ is known.

For FGMs, the material properties $k, \rho$ and $c$ are complex functions of spatial position $x$, and the
diffusion equation (1) is not amenable to analytical solutions. The numerical technique has gained wide acceptance in engineering applications. In Section 2.2, we introduce an FE/FD method to solve the above Eq. (1).

2.2. Finite element/finite difference method

The FE/FD method involves two essential procedures: (a) using finite element space discretization to obtain a first-order system of differential equations, and (b) finding transient response via finite difference technique.

2.2.1. (a) Finite element formulation

Suppose the medium undergoes a virtual temperature change $\delta T$, multiply Eq. (1) by $\delta T$ and then integrate it in the entire space domain $\Omega$, giving:

$$\int_\Omega \left[ \frac{\partial T}{\partial t} - \left( k_i \frac{\partial T}{\partial x_i} \right) - Q \right] \delta T \, dV = 0. \quad (3)$$

Since we assume that $\delta T$ is zero on the boundary $S_r$, from Eq. (2a) expression (3) becomes, after Green formula,

$$\int_\Omega \frac{\partial T}{\partial t} \delta T \, dV + \int_{S_q} \delta T \, dV - \int_{S_q} q \delta T \, dV = \int_\Omega Q \delta T \, dV = 0. \quad (4)$$

Now approximate the continuum by a finite number of elements. Let the continuum be divided into a finite number of elements interconnected only at nodal points. For each element occupying space $\Omega$, the temperature at any point can be expressed in terms of their values at their nodal points by:

$$T(x_1, x_2, x_3, t) = [N_0] \{ T \} \quad (5)$$

in which $[N_0]$ is known as the shape function matrix and is a function of spatial positions, $\{ T \}$ is a vector which contains the temperature values at the nodal points of the element. It follows from Eq. (5) that temperature gradients $\partial T/\partial x_i$ at any point in region $\Omega$ can be written as

$$\{ \partial T \} = \{ \partial T/\partial x_1, \partial T/\partial x_2, \partial T/\partial x_3 \}^T = [B_0] \{ T \}, \quad (6)$$

where $[B_0] = \[L]\{N_0\}$ and $[L]$ denotes a differential operator matrix. Substituting Eq. (6) into the thermal constitutive relations, we obtain the heat fluxes $q_i$ in the element:

$$\{ q \} = \{ q_1, q_2, q_3 \}^T = -[K][B_0] \{ T \}, \quad (7)$$

where $[K] = [k_{ij}]$ is a matrix containing the thermal conductivities of the medium.

Finally, by substituting Eqs. (5)–(7) into Eq. (4), the finite element approximation of the heat equation can be obtained as (after assembling)

$$[C_{00}] \{ \dot{T} \} + [K_{00}] \{ T \} = \{ p_0 \}, \quad (8)$$

where the dot represents differentiation with respect to time. The element matrices and external heat load vector are given by

$$[C_{00}] = \int_\Omega \rho c[N_0]^T[N_0] \, d\Omega, \quad (9a)$$

$$[K_{00}] = \int_\Omega [B_0]^T[K][B_0] \, d\Omega, \quad (9b)$$

$$\{ p_0 \} = -\int_S [N_0]^T \overline{\mathbf{T}} \, ds + \int_\Omega Q[N_0]^T \, d\Omega. \quad (9c)$$

In Eqs. (9), the material properties $\rho$, $c$ and $[K]$ are functions of spatial coordinates $x_i$, and $\{ T \}$ and $\{ p_0 \}$ are functions of time $t$. If the space is divided into a large number of elements, the material properties can be treated as constants in each element. If the applied thermal loads are independent of time, then $\{ p_0 \}$ is a constant vector. The numerical integration scheme, such as Gauss–Legendre integration, is used to evaluate the integrals involved in Eqs. (9).

The problem now is to solve the matrix differential equation (8). There are many general methods and several techniques for solving first-order matrix differential equations. Among many numerical techniques, the method of finite difference has been proven popular in finite element analysis.

2.2.2. (b) Finding transient response via finite difference

Since we cannot determine the nodal temperature $\{ T \}$ from Eq. (8) for all values of time $t$ in an interval $[0, t_0]$, we will have to be satisfied with computing approximations $\{ T \}_m$ of $\{ T(t_m) \}$ for some points $\{ t_m \}_{m=0}^M$ in the interval. We assume that the points are equidistant, i.e., that $t_m = m \Delta t$, $m = 0, \ldots, M$, where the step length $\Delta t$ is defined as $\Delta t = t_0/M$ for an integer $M$. Assuming that $\{ T_m \}$ has been known, then the approximation of $\{ T \}_{m+1}$ can be calculated from [13]:

$$\left( \frac{C}{\Delta t} + \frac{1}{2} [K] \right) \{ T \}_{m+1} = \left( \frac{C}{\Delta t} - \frac{1}{2} [K] \right) \{ T \}_m + \frac{1}{2} \{ p_m \} + \frac{1}{2} \{ p_{m+1} \}, \quad (10)$$

where $m = 0, \ldots, M$, $\{ p_m \} = \{ p(t_m) \}$, $\{ T \}_{m+1}$ on the left-hand side of Eq. (10) are unknowns, and all of the terms on the right-hand side are known. Eq. (10) represents a general family of recurrence relations, which is unconditionally stable and converges for the time portion with the truncation error of order $(\Delta t)^2$. Therefore, the number of time steps in Eq. (10) can be chosen such that the pre-required precise can be achieved.

In high temperature environments, material properties (density $\rho$, specific heat $c$ and thermal conductivity $k$, etc.) may become temperature-dependent. Finite element equations for such a problem can be derived using a similar procedure outlined in this section. The matrix
equations obtained have the same forms as those given there, provided that the coefficient matrices \([C]\) and \([K]\) are functions of temperature \(T\) and/or its gradients. The system equations will then become non-linear and can be solved iteratively. To avoid iterative operation, one can use the finite difference scheme with the additional assumption that material properties (matrices \([C]\) and \([K]\)) at time interval \((t_m, t_{m+1})\) are functions of the temperature vector \(\{T\}_m\) at time \(t_m\), which has already been known in each time step.

2.3. Explicit expressions for one-dimensional transient thermal conduction

One-dimensional transient thermal conduction is important for FGMs application in high temperature environment. Tanigawa et al. [1] and Jin and Paulino [7] used a laminated material model to solve the governing equation of heat conduction. They modeled the FGM by a laminated composite and each lamina was assumed as a homogeneous layer. The finite element formulation in the preceding section is established in the most general way. A practical situation arises where the system equations are one-dimensional plates, shells and spheres.

Referring to the coordinate systems shown in Fig. 1, the equations of one-dimensional heat conduction along the radial direction of a plate, a cylinder and a sphere can be written as:

\[
\rho \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 k(r) \frac{\partial T}{\partial r} \right) + Q,
\]

where \(L\) are 0, 1 and 2 for plate, cylinder and sphere, respectively, \(r\) is the radial coordinate. If the medium is divided into a number of linear interpolation elements along the \(r\)-direction (e.g., \(N\) elements and \(N+1\) nodes) with each element’s properties being assigned with constants, then the global matrices \([C]\) and \([K]\) can be obtained in closed-form.

We start from the \(i\)th element whose size along the \(r\)-direction is \(l_i\). Denote the material properties of the \(i\)th layer with a subscript \(i\). If the temperature inside the element is assumed a linear function of position \(r\), where \(r \in [r_i, r_{i+1}]\), then a linear shape function \([N]\) for the \(i\)th element can be chosen as:

\[
[N] = \begin{bmatrix}
1 - \frac{r - r_i}{l_i}
\end{bmatrix}.
\]

The “geometry matrix” \([B]\) is obtained from Eqs. (6) and (12) and is given by:

\[
[B]_i = [-1/l_i, 1/l_i].
\]

If we write the element matrices \([C]\), \([K]\) and \([p]\) for the \(i\)th elements as follows:

\[
[C]_i = \begin{bmatrix}
C_{11}^{(i)} \\
C_{12}^{(i)}
\end{bmatrix}, \quad [K]_i = \begin{bmatrix}
K_{11}^{(i)} & K_{12}^{(i)} \\
K_{12}^{(i)} & K_{22}^{(i)}
\end{bmatrix},
\]

\[
[p]_i = \begin{bmatrix}
p_1^{(i)} \\
p_2^{(i)}
\end{bmatrix},
\]

then the assembly of element matrices to form the global matrices in Eq. (8) gives:

(1) \(\{T\} = \{T_1, T_2, \ldots, T_{N+1}\}^T\).

(2) The non-zero \(i\)th row and \(j\)th column elements \(C_{ij}\) in the global matrix \([C]\) are:

- The first row: \(C_{11} = C_{11}^{(i)}, \quad C_{12} = C_{12}^{(i)}\).
- The \(i\)th row \((N+1 > i > 1)\): \(C_{ii} = C_{ii}^{(i-1)}, \quad C_{i1} = C_{i1}^{(i-1)} + C_{i1}^{(i-2)}\).
- The last row: \(C_{N+1} = C_{N+1}^{(N)}, \quad C_{N+1} = C_{N+2}^{(N)}\).

(3) The assembly of the global matrix \([K]\) is same as that of the global matrix \([C]\).

(4) The \(i\)th elements \(p_i\) in the global load vector \(\{p\}\) are:

- \(p_1 = p_1^{(i)}, \quad p_2 = p_2^{(i)} + p_2^{(i-1)}\).

\[
\text{Fig. 1. Boundary conditions and coordinate system of (a) a plate, and (b) a circular cylinder or a sphere.}
\]
If we further assume that materials in each element are homogeneous, the evaluation of Eqs. (9) would then give the following element matrices:

(i) Plate

The volume integral \(d\Omega\) for a straight plate is \(d\Omega = S_i d\tau\), where \(S_i\) is the area of the plate at the \(i\)th element (the area is in the plane perpendicular to the \(r\)-axis). The results are:

\[
[C]_i = S_i \int \rho c[N_0]^T[N_0] d\tau = \frac{\rho c}{6} \left[ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right] S_i, \tag{15a}
\]

\[
[K]_i = S_i \int \left[ B_0 \right]^T k[B_0] d\tau = \frac{k_i}{6} \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] S_i, \tag{15b}
\]

\[
[p]_i = \left\{ \frac{\mathbf{H}_i S_i}{-\mathbf{H}_i S_i} \right\} + S_i \int Q[N_0]^T d\tau. \tag{15c}
\]

(ii) Cylinder

The volume integral \(d\Omega\) for an axially symmetric cylinder is \(d\Omega = 2\pi r H_i d\tau\), where \(H_i\) is the length (along the axisymmetric axis) of the cylinder at the \(i\)th element. Hence,

\[
[C]_i = 2\pi H_i \int \rho c[N_0]^T[N_0] r d\tau
= \frac{\pi \rho c}{6} \left[ \begin{array}{cc} r_{i+1}^2 + 3r_i & r_{i+1} + r_i \\ r_{i+1} + r_i & r_{i+1} + r_i \end{array} \right] H_i, \tag{16a}
\]

\[
[K]_i = 2\pi H_i \int \left[ B_0 \right]^T[k][B_0] r d\tau
= \pi k_i \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] H_i, \tag{16b}
\]

\[
[p]_i = \left\{ \frac{2\pi \mathbf{H}_i H_i r_i}{-2\pi \mathbf{H}_i H_i r_i} \right\} + 2\pi H_i \int Q[N_0]^T r d\tau. \tag{16c}
\]

(iii) Sphere

The volume integral \(d\Omega\) for a rotationally symmetric sphere is \(d\Omega = 4\pi r^2 d\tau\). Then,

\[
[C]_i = 4\pi \int \rho c[N_0]^T[N_0] r^2 d\tau
= \frac{\pi \rho c}{15} \left[ \begin{array}{cc} 2r_{i+1}^2 + 6r_i + r_{i+1}^2 + 12r_i^2 & 3r_{i+1}^2 + 4r_i r_{i+1} + 3r_i^2 \\ 3r_{i+1}^2 + 4r_i r_{i+1} + 3r_i^2 & 12r_{i+1}^2 + 6r_i r_{i+1} + 2r_i^2 \end{array} \right], \tag{17a}
\]

\[
[K]_i = 4\pi \int \left[ B_0 \right]^T[k][B_0] r^2 d\tau
= \frac{4\pi r_{i+1}^2 + r_i r_{i+1} + r_i^2}{3} k_i \left[ \begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array} \right], \tag{17b}
\]

\[
[p]_i = \left\{ \frac{4\pi \mathbf{H}_i r_{i+1}^2}{-4\pi \mathbf{H}_i r_{i+1}^2} \right\} + 4\pi \int Q[N_0]^T r^2 d\tau. \tag{17c}
\]

In expressions (15)–(17) \(r_b = r_{N+1} > r_N > \cdots > r_2 > r_1 = r_a, l_i = r_{i+1} - r_i\).

Now that all matrices are known, we can write a computation program to find the transient solution of Eq. (8), using the finite difference scheme given in Sub-section 2.2(b). Although material properties in each element are assumed as constants, we can improve the computational accuracy by increasing the number of elements. The above discretization method is in a way similar to the laminated plate model, used by Tanigawa et al. [1] and Jin and Paulino [7], to solve the one-dimensional transient heat conduction equations. In those analyses, a FGM was divided into a number of layers, with each layer being treated as a homogeneous layer.

2.4. Thermal stresses

The preceding sections established the transient temperature solution. After the temperature is determined, the stress analysis is relatively straightforward. As an example, we establish the stress field in cylinders (hollow or solid), using the above finite element model.

We start from the \(i\)th element, which is an axially symmetric cylinder. Suppose the cylinder is under a plane strain status, the displacement and stress along the radial direction inside the element can be found from thermal stress textbooks [see, for example, Ref. [14]] as:

\[
u_t(r) = \frac{1}{1 - \nu_i} \frac{\alpha_i}{r} \int_{r_i}^{r} T_i(r) r d\tau + \frac{M_{i+1} - M_{i}}{r}, \tag{18a}
\]

and

\[
\sigma_t(r) = \frac{E_i}{1 - \nu_i} \left( -\frac{1}{r} \frac{\nu_i}{r^2} \int_{r_i}^{r} T_i(r) r d\tau + \frac{M_{i+1}}{1 - 2\nu_i} - \frac{M_{i}}{r} \right), \tag{18b}
\]

respectively, where \(E\) is Young's modulus, \(\nu\) is Poisson's ratio, and \(\alpha\) is thermal expansion coefficient, \(M_{i+1}\) and \(M_i\) are unknown constants. Hereafter, the subscript \(i\) represents the \(i\)th element, and the subscripts \(I\) and \((I + 1)\) denote, respectively, the inner and outer surfaces of the \(i\)th element.

Inside each element, the temperature \(T_i(r)\) is assumed to vary linearly from \(T_I\) on its inner surface to \(T_{I+1}\) on its outer surface. Thus, on the outer and inner surfaces of the \(i\)th element, Eqs. (18) become:

\[
u_t(r_{i+1}) = \frac{1}{6} \frac{\nu_i}{r_{i+1}} \frac{r_{i+1} + 2r_i}{6} \frac{r_{i+1} + r_i}{6} T_I + \frac{2}{6} \frac{r_{i+1} + r_i}{6} T_{I+1} \frac{1}{l_i}, \tag{19a}
\]

and

\[
\sigma_t(r_{i+1}) = \frac{E_i}{6} \frac{1}{1 - \nu_i} \left( -\frac{1}{r} \frac{\nu_i}{r^2} \frac{r_{i+1} + 2r_i}{6} \frac{r_{i+1} + r_i}{6} T_I + \frac{2}{6} \frac{r_{i+1} + r_i}{6} T_{I+1} \frac{1}{l_i} \right) \times \left( -\frac{1}{r} \frac{\nu_i}{r^2} \frac{r_{i+1} + 2r_i}{6} \frac{r_{i+1} + r_i}{6} T_I + \frac{2}{6} \frac{r_{i+1} + r_i}{6} T_{I+1} \frac{1}{l_i} \right), \tag{19b}
\]
and

\[ u_i(r) = M_i r_i + \frac{M_{2i}}{r_i}, \quad (20a) \]

\[ \sigma_{ir}(r) = \frac{E_i}{1 + v_i} \left( \frac{M_{1i}}{1 - 2v_i} \right) \frac{M_{2i}}{r_i^2}, \quad (20b) \]

in which \( i = 1, \ldots, N - 1 \) and \( N \) is the number of elements. The displacement and stress obtained from Eqs. (19) and (20) should be continuous at the \( i \)th interface at \( r = r_{i+1} \). This gives:

\[ \begin{align*}
1 + v_i & \frac{x_i}{r_{i+1}} (r_{i+1} + 2r_i) T_i + (2r_{i+1} + r_i) T_{i+1} I_i + M_i r_{i+1} + \frac{M_{2i}}{r_{i+1}} \\
& = M_{1(i+1)} r_{i+1} + M_{2(i+1)} r_{i+1}^{-1}, \\
\frac{E_i}{1 + v_i} & \left( \frac{1}{1 - 2v_i} \right) \\
& = E_i + M_{1i} + \frac{M_{2i}}{r_{i+1}^2} - M_{2(i+1)} r_{i+1}^{-2}, \\
& = \left( \frac{M_{1(i+1)} - M_{2(i+1)}}{r_{i+1}^2} \right), \\
& = \left( \frac{M_{1(i+1)} - M_{2(i+1)}}{r_{i+1}^2} \right), \\
& = \left( \frac{M_{1(i+1)} - M_{2(i+1)}}{r_{i+1}^2} \right), \\
& = \left( \frac{M_{1(i+1)} - M_{2(i+1)}}{r_{i+1}^2} \right).
\end{align*} \]

where \( i = 1, \ldots, N - 1 \). There are \( 2(N - 1) \) equations in Eqs. (21). Further, at the inner surface of the cylinder \( \sigma_{r1} \) is zero (for a solid cylinder, \( \sigma_{r1} \) does not necessary vanish but it must be finite, therefore \( M_{2(i+1)} = 0 \)), and at the outer surface of the cylinder \( \sigma_{r(N+1)} \) is zero. Hence, the total number of continuity and boundary conditions is \( 2N \), which can be used to determine \( 2N \) unknown constants \( M_{1i} \) and \( M_{2i} \) \( (i = 1, \ldots, N) \). Once those constants are determined, the radial displacement \( u_r \) and stress \( \sigma_{r} \) can be calculated from Eqs. (18). The circumferential stress, \( \sigma_{\theta} \), can be obtained as

\[ \sigma_{\theta}(r) = \frac{E_i}{1 - v_i} \frac{x_i}{r^2} \int_0^r T_i(r') r' dr' + \frac{E_i M_{1i}}{(1 + v_i)(1 - 2v_i)} \left( \frac{M_{2i}}{r} \right) + \frac{E_i M_{2i}}{1 + v_i} \left( \frac{M_{2i}}{r} \right). \]

This completes the stress distribution in a non-homogeneous shell. To further demonstrate the application of the FE/FD method in non-homogeneous materials, we investigate the thermal shock resistance of a FGM plate in Section 3.

3. Thermal shock of a FGM

Consider an infinitely long plate of thickness \( h \) as shown in Fig. 2. The plate is initially at a constant temperature. Without loss of generality, the initial constant temperature can be assumed as zero. The surfaces \( y = 0 \) and \( y = h \) of the plate are suddenly cooled down to a temperature \( T_0 \). Since the heat will flow only in the \( y \)-direction, the initial and boundary conditions for the temperature field are

\[ T = 0, \quad t = 0, \quad (23) \]

\[ T = -T_0, \quad y = 0, \quad (24) \]

\[ T = -T_0, \quad y = h. \quad (25) \]

Here an idealized thermal shock boundary condition is assumed, i.e., the heat transfer coefficient on the surfaces of the FGM plate is infinitely large. This is the most severe thermal shock load on the plate, which is important for the study of thermal stresses in a quenched specimen [15]. For such a transient thermal conduction problem, the temperature field can be readily obtained from the proposed FE/FD method, as outline in Section 2, and shown in Fig. 3 as an example. According to [16], the transient thermal stress in a fully free FGM plate is:

\[ \sigma_{\theta}(y) = \frac{E(y)}{1 - v(y)} [A\theta + B - \alpha(y)(T'(y) - T_0)], \]

where \( E \) and \( v \) are elastic modulus and Poisson’s ratio, respectively, \( \alpha \) is the thermal expansion coefficient. The constants \( A \) and \( B \) are determined from \( \int_0^h \sigma_{\theta}(y) dy = 0 \) and \( \int_0^h \sigma_{\theta}(y) y dy = 0 \).

Fig. 2. An FGM plate subjected to a thermal shock on its surfaces.

Fig. 3. Model verification: one-dimensional transient temperature distribution in a homogeneous layer (\( M \): number of elements, \( t_0 = \rho c h^2 / k \)).
3.1. Property distribution of a functionally graded material

A two-phase ceramic/metal FGM is considered here. The FGM is a combination of metal Ni and ceramic TiC. The thermophysical properties of Ni and TiC are listed in Tables 1 and 2. In order to predict the properties of the FGM, it is important to recognize that FGMs are basically composite materials. The volume fraction of ceramic in the FGM is expressed as a power function of \( y \):

\[
V_c = 1 - (y/h)^g
\]

(27)

in which \( V_c \) represents the volume fraction of the ceramic phase in the FGM, \( g \) is known as the gradient index. Here \( y = 0 \) corresponds to pure ceramic and \( y = h \) to pure metal.

To evaluate the properties of the FGM, micromechanics models of composite materials have to be used. The self-consistent method (SCM) has several features which make it attractive in FGM problems. Zuiker [17] pointed out that, the SCM estimates provide a simple, initial estimate of effective properties which may be used to relate optimal property distributions to the required volume fraction and, presumably, processing requirements. Effective properties for reinforcements in a continuous matrix are calculated from

\[
V_c = \frac{3K_m}{3K_m + 4\mu_0} + \frac{5\mu_c}{5\mu_c - 3K_c} - \frac{5\mu_m}{5\mu_m - 3K_m},
\]

\[
K_0 = \frac{(3K_c + 4\mu_0)(3K_m + 4\mu_0)}{3V_c(3K_m + 4\mu_0) + 3(1 - V_c)(3K_c + 4\mu_0)} - \frac{4}{3}\mu_0,
\]

\[
z_0 = \frac{z_c - z_m}{1/K_0 - 1/K_m},
\]

\[
\frac{(1 - V_c)(k_m - k_0)}{k_m + 2k_0} + \frac{V_c(k_c - k_0)}{k_c + 2k_0} = 0,
\]

(28)

where \( V_c \) is the volume fraction of ceramic, \( K \) the bulk modulus, \( \mu \) the shear modulus, \( x \) the thermal expansion coefficient, \( k \) the thermal conductivity, and subscripts ‘c’, ‘m’, and 0 represent particular phase, matrix phase, and functionally graded composite material, respectively. The principal advantage of the SCM is that the property estimates are independent of which phase is the reinforcing particulate and which is the continuous matrix. This is particular for FGM whose volume fraction of the constituent phases varies in a wide region (usually from 0% to 100%).

Other properties, including the specific heat and the density of the FGM are estimated simply from the rule of mixtures. The fracture toughness of the FGM is a very complicated function of the volume fractions of the constituent phases, and is preferably determined from experiments. As a simple, but may be very rough approximation, we also use the rule of mixtures to determine the fracture toughness of the FGM.

3.2. Thermal stress distribution

In the following analysis, thermal stress and time scale are normalized, respectively, by \( \sigma_0 \) and \( t_0 \), where \( \sigma_0 = E_c z_c T_0/(1 - \nu_c) \) and \( t_0 = \rho_c h^2/k_c \). Again, subscript ‘c’ represents the ceramic TiC. The dimensionless stress \( \sigma/\sigma_0 \) is plotted against the non-dimensional location \( y/h \) in Figs. 4–6 at different time \( t/t_0 \). It is clear that if \( T_0 > 0 \) (cold shock, see Eqs. (24) and (25)), the region near the surface of the plate is in tension while a compressive zone is developed at the plate center. The results shown in these figures also suggest that the material nonhomogeneity (i.e., the gradient index) has a pronounced influence on the thermal stresses. Further, the overall thermal stress level in the medium decreases as time increases. The maximum tensile stress is attained at the surface while the compressive stress is largest near the center of the plate. It is found that a maximum value of

\[
\sigma_{\text{max}} = E_c z_c T_0/(1 - \nu_c)
\]

(29)

is achieved at the surface of the plate at beginning of the thermal shock, i.e., at \( t = 0 \).

### Table 1
Thermo-physical properties of the model materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Specific heat (J/kg K)</th>
<th>Thermal conductivity (W/m K)</th>
<th>Coefficient of thermal expansion ( (10^{-6} / \text{C}) )</th>
<th>Density ( (g/cm^3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TiC</td>
<td>( c_v ): 134</td>
<td>( k_v ): 25.1</td>
<td>( \alpha_v ): 7.4</td>
<td>( \rho_v ): 4.94</td>
</tr>
<tr>
<td>Ni</td>
<td>( c_w ): 439.5</td>
<td>( k_w ): 90.5</td>
<td>( \alpha_w ): 13.3</td>
<td>( \rho_w ): 8.89</td>
</tr>
</tbody>
</table>

### Table 2
Mechanical properties of the model materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
<th>Fracture strength (MPa)</th>
<th>Fracture toughness (MPa m(^{1/2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>TiC</td>
<td>( E_v ): 320</td>
<td>( \nu_v ): 0.195</td>
<td>( \sigma_{\text{fr}} ): 230</td>
<td>( K_v ): 3</td>
</tr>
<tr>
<td>Ni</td>
<td>( E_w ): 206</td>
<td>( \nu_w ): 0.312</td>
<td>( \sigma_{\text{fr}} ): 332</td>
<td>( K_{\text{fr}} ): 100</td>
</tr>
</tbody>
</table>
3.3. Cracking due to thermal shock

As described above, the surface of the plate undergoes a transient tensile stress under a cold shock, while the center of the plate has the largest tensile stress under a hot shock. It is then expected that a crack may initiate at the ceramic surface under cold shocking. In fact, it has been experimentally observed that surface cracking in FGMs is the most common failure mode of a metal-ceramic FGM when it is subjected to a thermal shock. The problem can be synthetized from the general solution for an edge crack in a finite plate, with the crack tip located at \( y = c \), as shown in Fig. 7. Since the crack plane is normal to the surface of the plate, it does not perturb the transient temperature distribution in this arrangement, determination of the temperature distribution would be quite straightforward by the proposed FE/FD method. The resulting thermal stress, obtained from Eq. (26), with opposite sign, becomes the crack surface tractions in the quasi-static, mixed boundary-value problem. Assuming \( x = 0 \) is a plane of symmetry, the homogeneous boundary conditions used in formulating the mixed problem are given by \( \tau_{yy}(x,0) = 0 \), \( \sigma_{yy}(x,0) = 0 \), \( \tau_{yy}(x,h) = 0 \), \( \sigma_{yy}(x,h) = 0 \). The additional mixed boundary conditions along the \( x = 0 \) plane are

\[
\begin{align*}
    u(0,y) &= 0, \quad y \notin (0,c), \\
    \sigma_{xx}(0,y) &= \sigma_0(y), \quad y \in (0,c),
\end{align*}
\]

where \( u \) is the \( x \) direction component of the displacement vector, \( \sigma_0(y,t) \) is the equal and opposite value of \( \sigma_{yy}(y,t) \) given by Eq. (26). The solution for an edge crack in a FGM with arbitrarily varied material properties has been developed in [18] by using a multi-layered plate model. In the analysis, the FGM region is treated as a number of thin layers (say \( N \) layers). The material properties are considered to be constants for each layer. The plane problem requires the solution of the following equilibrium equations for each layer:

\[
\begin{align*}
    (\eta_j - 1) \nabla^2 u_j + 2 \left( \frac{\partial^2 u_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial x \partial y} \right) &= 0, \\
    (\eta_j - 1) \nabla^2 v_j + 2 \left( \frac{\partial^2 u_j}{\partial x \partial y} + \frac{\partial^2 v_j}{\partial y^2} \right) &= 0,
\end{align*}
\]

Fig. 7. An edge crack in a FGM plate.
where \((J = 1, \ldots, N)\), \(u, v\) are the \(x, y\) components of the displacement vector, \(\eta(y) = 3 - 4\hat{\nu}(y)\) for plane strain and \(\eta(y) = [3 - \nu(y)]/[1 + \nu(y)]\) for plane stress, where \(\nu(y)\) is the locally varying Poisson’s ratio.

The solution of the equations of elasticity subjected to the conditions specified by Eqs. (30) results in an integral equation for a crack subjected to relatively general loading conditions. Algebraic details concerning derivation of the integral equation for the crack problem have been given in [18]. The resulting singular integral equation has the form

\[
\frac{1}{4\pi} \int_0^{\infty} \frac{g(r)dr}{r - y} + \int_0^c A(r, r)g(r)dr = \sigma_0(y, t),
\]

(32)

where \(y \in (0, c)\), subscript \(J\) stands for the \(J\)th layer, \(\mu_j\) is the shear modulus, \(A_j\) is a Fredholm kernel, and \(g(y)\) is a displacement discontinuity function along the cracked plane defined by \(g(y) = \partial u(0, y)/\partial y\). Since time \(t\) only enters the problem through \(\sigma_0(y, t)\) in Eq. (32), the singular integral equation is solved separately for individual values of \(t\). To this end, the solution of Eq. (32) can be written as [16,18]:

\[
g(r) = \frac{1}{\sqrt{2(1 - r/c)}} \sum_{n=0}^{\infty} a_n T_n(2r/c - 1),
\]

(33)

where \(T_n\) is the Chebyshev polynomial of the first kind, and \(a_0, a_1, \ldots\) are unknown constants. After substituting Eq. (33) into Eq. (32), truncated with the first \(M\) terms, and by using a collocation technique, the unknown constants \(a_0, a_1, \ldots\) can be obtained. The SIF \(K\) at the crack tip at \(y = c\) can be defined and calculated as

\[
K(c) = \lim_{y \to c^+} \sqrt{2\pi(y - c)} \sigma_{y} = -\frac{4\mu(c)}{1 + \eta(c)} \sqrt{\pi c} \sum_{n=0}^{M} a_n.
\]

(34)

Consider again the infinite plate shown in Fig. 2 subjected to a cold shock. The problem is idealized to the case of a single-edge cracked plate, as shown in Fig. 7, and the SIF \(K\) is calculated for a cold shock event. The normalized SIF \(K/k_0\), where \(k_0 = E_0\varepsilon_0 T_0/ \sqrt{\pi h} / (1 - \nu_k)\), is plotted against dimensionless time \(t/k_0\) for different gradient indices in Figs. 8–10 for selected normalized crack depth \(c/h\), where \(c\) is the crack depth. For any given \(c/h\), the stress intensity increases from an initial zero value, displays a peak value and then decreases as time approaches infinity. This means that the maximum thermal SIF occurs only at transient state. The magnitude of \(K\) depends on crack length, and achieves a peak value at a certain crack depth. It is further noted that as the gradient index increases, the overall level of the SIF does not change significantly. Therefore, the transient thermal SIF for a FGM layer with a higher metal content is similar to those with a lower metal content. Although the SIF is
not decreased by the increasing metal content, it will be
seen that the strength of the FGM layer with high metal
contents is much higher than that of the FGM with
lower metal contents. To illustrate this, it is more con-
venient to introduce a stress intensity factor ratio
(SIFR):

\[ f = \frac{K(c)}{K_{IC}(c)}, \] (35)

where, \( K \) is the local SIF, and \( K_{IC} \) is the local fracture
toughness in the FGM layer. Both of them are functions
of crack tip position \( y = c \). Figs. 11–13 plot the variation
of normalized strength ratio \( \tilde{f} = f/f_0 \) with time, at se-
lected crack lengths, where

\[ f_0 = \frac{E_c x_c T_0 \sqrt{\pi h}}{K_{IC}(\text{ceramic})}. \] (36)

In Eq. (36), \( h \) is the thickness of the medium. Note
that \( f_0 \) and \( f \) are different: \( f_0 \) is a constant but \( f \) is a
function of crack tip location \( c \) since both \( K(c) \) and
\( K_{IC}(c) \) vary with \( c \). It is clear from Eqs. (27) and (35)
that as the gradient index increases, the strength ratio \( f \)
also increases. Noting that a high value of \( f \) corresponds
to a lower fracture strength margin, and \( f \) equals 1
means that the SIF has achieved the local fracture
toughness of the medium. Hence, the strength of the
FGM layer can be increased by increasing the metal
content. Further, note from Figs. 11–13, if we consider a
crack of a given initial length, we see that the temporal
behavior is alike for all crack depths. The envelope of
this family of curves is the locus of the highest SIFs
reached for every crack depth at any time. Envelops for
gradient indexes \( g = 0.5, 1, 2 \) are displayed in Fig. 14. It
can be seen that if the content of metal is small (e.g., for
\( g = 2 \)), with crack growth from near the ceramic-rich
region (near \( c = 0 \)) toward the metal rich region (near
\( c = h \)), the strength of the FGM will first decrease and
then increase. Conversely, for medium and high values
of metal volume fraction (e.g., for \( g = 1 \) and \( g = 0.5 \),
the strength will always increase with crack length. This
means that the crack will not grow, showing consider-
able crack growth resistance behavior.

3.4. Thermal shock resistance

Thermal shock resistance is a major issue in the de-
sign of engineering materials for high-temperature ap-
plications. A central problem in designing against
thermal shock is the identification of an appropriate
material selection criterion in order to select the most
shock-resistant material for a given application. Gen-
erally, there are two distinct criteria for the determina-
tion of thermal shock resistance: the stress-based
criterion and the fracture mechanics-based criterion.
A stress-based failure criterion for cold shock is that the maximum thermal stress that appears on the plate surface \( \sigma_{\text{max}}(t) \) is equal to or exceeds the strength limit of the ceramic \( \sigma_{\text{bc}} \). The maximum temperature sustainable by \( T_c \) follows from Eq. (29) as

\[
T_c = T_c^{\text{stress}} = \frac{\sigma_{\text{bc}}}{E_c \zeta_c/(1 - \nu_c)}. \tag{37}
\]

Similarly, for the fracture-based failure criterion, the maximum thermal SIF \( K_{\text{max}}(c, t) \) must equal the local fracture toughness of the material \( K_{IC}(c) \). It is clear that the maximum temperature sustainable by the medium \( T_c \) depends on the depth of the pre-existing crack, because the maximum SIF depends on the crack depth. Since thermal SIFs are zero at \( c = 0 \) and \( c = h \), the failure of the plate should be predicted by fracture mechanics-controlled failure and for the stress-based failure criterion. By equating Eqs. (37) and (38), this depth \( c_1 \) can be determined from the relation

\[
K_{IC}(\text{ceramic}) \left( \frac{\sigma_{\text{bc}} \sqrt{\pi h}}{E_c \zeta_c/(1 - \nu_c)} \right) = T_c^{\text{stress}}. \tag{40}
\]

As \( f(c) \) is a complex function of the crack depth, \( c_1 \) can only be found numerically. It can be shown from Fig. 15, except at \( c_{\text{min}} \), there are two roots in Eq. (40), namely, \( (c_1, c_2) \) for \( c_1 \).

Consideration of Eqs. (37) and (38) reveals that the admissible temperature for thermal shock is less for the fracture mechanics-based criterion than for the stress-based criterion, at a sufficiently large plate thickness. A transient plate thickness value \( h_t \) exists for which \( T_c^{\text{min}} \) is equal for fracture mechanics-based criteria and stress-based criterion. Then, upon equating \( T_c^{\text{min}} \) values for the stress-based criterion (37) and for the fracture mechanics-based criterion (38), it is found that

\[
h_t = 14.0 \left( \frac{K_{IC}(\text{ceramic})}{\sigma_{\text{bc}}} \right)^2. \tag{41}
\]

from infinity to a minimum and then increases again to infinity. The crack depth at which \( T_c \) is minimized is denoted by \( c_{\text{min}} \). The minimum temperature sustainable by the ceramic in this example is:

\[
T_c^{\text{min}} = 6.62 \frac{K_{IC}(\text{ceramic})}{E_c \zeta_c/(1 - \nu_c)} \tag{39}
\]
Quantitatively, for the material system considered, Eq. (41) gives:

\[ h_t = 2.4 \text{ mm} \]  \hspace{1cm} (42)

Note that these expressions are only valid for the material parameters used in the numerical analysis. Therefore, the thermal shock resistance of a FGM plate with thickness smaller than \( h_t \) will be controlled by the stress-based criterion; a plate with thickness larger than \( h_t \) will be controlled by the fracture mechanics-based criterion. Thus, a FGM plate whose thickness is \( h_t \) has the same thermal shock resistance according to the stress criterion and the fracture criterion. The fracture mechanics-based criterion is conservative for a plate whose thickness is above \( h_t \), the stress-based criterion is conservative for a plate whose thickness is below \( h_t \).

4. Conclusion

Knowledge of the temperature distribution and associated thermal stresses in FGMs under severe thermal environment is important in order to evaluate their integrity. The one-dimensional transient temperature field is determined from using the finite element method together with the finite difference technique. Closed form expressions for a one-dimensional plate, cylinder, and sphere elements are given.

The thermal shock behavior of a FGM has been investigated theoretically for a plate containing a surface crack. The surfaces of the plate undergo sudden cooling. The temperature and stress histories in the plate are given. The stress field is tensile near the surfaces of the plate and gives rise to a stress-intensity factor for a pre-existing crack. The maximum thermal stress factors occur at transient state.

The thermal shock strength of a FGM is evaluated. The maximum temperature that the material can sustain without catastrophic failure is analyzed according to the maximum local tensile stress criterion and the maximum stress intensity factor criterion. It is found that an FGM with high metal contents exhibits significant resistance to crack growth from the ceramic side to the metal side. A transient plate thickness value \( h_t \) was obtained for which the admissible temperature is equal for fracture mechanics-controlled and stress-controlled failure criteria. The thermal shock resistance of a FGM plate with thickness smaller than \( h_t \) will be controlled by the stress-based criterion, and a plate with thickness larger than \( h_t \) will be controlled by the fracture mechanics-based criterion.

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