Strength distributions of warm frozen clay and its stochastic damage constitutive model

Yuanming Lai *, Shuangyang Li, Jilin Qi, Zhihua Gao, Xiaoxiao Chang

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Abstract

There are many defects, such as fissures and cavities, in warm frozen clay and in warm ice-rich frozen clay. These defects are distributed randomly, which make some mechanical properties of these clays exhibit great uncertainty. Thus it is unreasonable to take some deterministic values as the mechanics parameters of these frozen clays. Furthermore, warm frozen clay and warm ice-rich frozen clay are less stable than clays at colder temperatures. Therefore, it is necessary to determine the strength and constitutive relationship of these clays using probabilistic methods. In this paper, based on the characteristics of the random distribution of defects in the warm frozen clay and warm ice-rich frozen clay, the strength distribution laws for both clays are investigated at −0.5 °C, −1.0 °C and −2.0 °C through a series of experimental data, respectively. The investigated results show that the Weibull distribution can more closely reflect the strength distribution law than other probability distributions. Based on the results mentioned above, a stochastic damage model for the warm frozen clay and warm ice-rich frozen clay was developed using the continuous damage theory and probability, as well as statistic theory. Comparing theoretical results of the model with experimental data at these three temperatures, respectively, it is found that the stress-strain relationships of the two kinds of frozen clays, especially their deformation characteristics after failures, could be described by the model very well. Since the range of strength variation for both kinds of frozen clays is very large, it is unsafe and unreasonable for engineering design to use the average strengths of frozen clay soils. The reliability of the strength of these two clays was investigated and discussed on the basis of the fact that the strength of warm frozen clay and warm ice-rich frozen clay satisfy the Weibull distribution.

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Keywords: Warm frozen clay; Warm ice-rich frozen clay; Strength distribution; Stochastic damage constitutive model

1. Introduction

The Qinghai–Tibetan Railway (QTR) is about 1142 km long, of which 275 km is underlain by warm permafrost (mean annual ground temperatures range from 0 to −1.5 °C), and 134 km is underlain by warm ice-rich permafrost (mean annual ground temperatures range from 0 to −1.5 °C and volumetric ice content is greater than 25%) (Cheng, 2003). The warm frozen and warm ice-rich frozen soils are composed of soil, ice, unfrozen water and air. Since there is ice in the frozen soil, the strength and deformation will greatly change with minor temperature variations. From field experiments, Ma (2006) found that large deformations in warm ice-rich frozen soil would occur even if the load is small, which may be the main reason for the differential settlement and many cracks along the subgrade of the QTR during the last 5 years. It is therefore urgent and important to look into the strength and deformation laws of warm frozen soil and warm ice-rich...
frozen soil. Previously, because of the complexity of the frozen soils and the limitation of the experimental conditions, a large number of experiments and research were performed only under low temperature conditions. There are few papers concerning warm frozen soil or warm ice-rich frozen soil. Ma (2006) studied the uniaxial compression strengths of warm ice-rich frozen clay at different water contents (20%, 40%, 60%, 90% and 120%) and found that the stress-strain relationship of warm ice-rich frozen clay is of a strain hardening manner and the form of failure is plastic. At the same time, the compressive strength of warm ice-rich frozen clay with water contents of 40%–90% increases linearly with decrease in temperature. Subsequently, Ma et al. (2007) studied the creep laws of warm ice-rich frozen soil at several test temperatures and proposed a creep equation. These conclusions were obtained using the deterministic method and were based on a small number of experiments. However, the structure of frozen soil varies with the random distribution of internal defects. Therefore, its mechanics properties exhibit randomness and uncertainty, and the stress-strain relationship, especially of warm frozen soil and warm ice-rich frozen soil, can not be described well deterministically. After comparing study, the Weibull distribution, among the many potential distribution laws, was chosen to solve these problems. A vast amount of uniaxial stress-strain experimental data, under different temperatures (–0.5 °C, –1.0 °C and –2.0 °C), was obtained on warm frozen clay and warm ice-rich frozen clay by testing. Then, a stochastic damage constitutive model was developed using the continuous damage theory and probability method, as well as statistical theory. Comparisons with the experimental data show that the whole stress-strain process could be well described by this model.

Since the randomness of strengths of warm frozen clay and warm ice-rich frozen clay is very large, it is unsafe and unreasonable to design engineering using average strengths. However, this problem could be solved if the probabilistic method is introduced into the design. Therefore, the reliability of the strengths of the two clays is investigated and discussed on the basis of the fact that the strengths of warm frozen clay and warm ice-rich frozen clay satisfy the Weibull distribution.

2. Experimental conditions and results

The soil specimens were made of clay brought from the Beiluhe site along the Qinghai-Tibetan railway and mixed with the desired amount of water. The clay is mainly montmorillonite and its main ingredients are SiO₂, Al₂O₃, Fe₂O₃, MgO, K₂O and TiO₂ in sequence according to their mass contents. The particle fractions are listed in Table 1. It can be seen that the main components were coarse-grained and silty clay. The frozen temperature of this kind of soil is –0.3 °C. The samples had an average dry unit weight of 18.1 kN/m³. Two groups of warm frozen clay and warm ice-rich frozen clay specimens with water contents of 17.6% and 30.0%, respectively, were made, 61.8 mm in diameter and 125.0 mm in height. Then, the samples are saturated in a sealed mold for 48 h under negative temperature environments (–0.5 °C, –1.0 °C and –2.0 °C). In order to ensure the reliability of the test results, 50 specimens for each group of soil were prepared. The specimens were placed in a MTS low temperature testing machine for 24 h at a given temperature. According to the Chinese code for soil test, the

<table>
<thead>
<tr>
<th>Particle sizes and their percentages (%)</th>
<th>&lt;0.0001 mm</th>
<th>0.0001–0.001 mm</th>
<th>0.001–0.005 mm</th>
<th>0.005–0.05 mm</th>
<th>0.05–2 mm</th>
<th>&gt;2 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12.42</td>
<td>41.18</td>
<td>44.65</td>
<td>1.75</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1. The specimen of warm ice-rich frozen clay.
Fig. 2. Stress–strain relationships of warm frozen clay at different temperatures.
Fig. 3. Stress–strain relationships of warm ice-rich frozen clay at different temperatures.
axial pressure was applied to the specimens until failure with the displacement rate of 1.25 mm/min, which is equal to the standard strain rate of 1.7 × 10^{-4}/s (The Professional Standards Compilation Group of People's Republic of China, 1999), at −0.5 °C, −1.0 °C and −2.0 °C with a precision of ±0.1 °C (As shown in Fig. 1), respectively.

Fig. 2 shows the stress-strain relationships of warm frozen clay at different temperatures. From this figure, we find that, under the three different temperature conditions, the stress-strain relationships of the warm frozen clay all show a strain softening. The curves of each group are similar at the start of loading. However, with increasing strain, the curves separated as defects developed randomly. Especially, after the warm frozen clays were damaged, the discreteness of these curves is larger than the before. The stress standard variances of warm frozen clay at strains 1.0%, 3.0% and 6.0% for the temperature −0.5 °C are 0.043 MPa, 0.048 MPa and 0.091 MPa, respectively. When the temperature of warm frozen clay is −1.0 °C, its stress standard variances at same strains mentioned above are 0.089 MPa, 0.147 MPa and 0.235 MPa, respectively. The corresponding stress standard variances of warm frozen clay for the temperature −2.0 °C are 0.101 MPa, 0.140 MPa and 0.235 MPa, respectively. Obviously, the standard variances of average stresses become larger with increasing strain, i.e., after the loading is applied on these specimens, the tiny defects in warm frozen clay will gradually develop, correspondingly, the discreteness of these curves becomes larger and larger. Since the discreteness of these curves is very large, the mechanical properties cannot be well described deterministically.

Similarly, Fig. 3 shows the stress-strain relationship of warm ice-rich frozen clay at different three testing temperatures. As opposed to warm frozen clay, the warm ice-rich samples exhibit some elastic-plastic characteristics instead of strain softening. From Fig. 3(a, b, c), we find that, when the strain is more than 2%, the discreteness of stress-strain relationship is large. The stress standard variances of warm ice-rich frozen clay at strains 1.0%, 6.0% and 12.0% for the temperature −0.5 °C are 0.051 MPa, 0.059 MPa and 0.058 MPa, respectively. When the temperature of warm ice-rich frozen clay is −1.0 °C, its stress standard variances at same strains mentioned above are 0.090 MPa, 0.092 MPa and 0.089 MPa, respectively. The corresponding stress standard variances at the temperature −2.0 °C are 0.130 MPa, 0.095 MPa and 0.081 MPa, respectively. From Fig. 3 and the data mentioned above, we can see that after the warm ice-rich frozen clay yields, the average stresses and their standard variances are relatively close. Because the standard variances are large, it is unreasonable to determine the mechanical properties of warm ice-rich frozen clay by the deterministic method. The strength distributions and stochastic constitutive relationships of warm frozen clay and warm ice-rich frozen clay will therefore be discussed and analyzed in detail.

The discreteness of stress—strain relationships of the two groups of frozen clay is large, and their strength variations are wide (as shown in Table 2). We find that, when the temperature of warm frozen clay is −1.0 °C, the strength variation range of frozen clay is the widest among these values, at 0.814 MPa. Since the strength randomness of the warm frozen clay and warm ice-rich frozen clay is very large, we will try to find which probability distribution it satisfies.

3. Strength distribution laws

Substituting the strengths of warm frozen clay and warm ice-rich frozen clay into several probability distributions, such as the Normal distribution, the Lognormal distribution, the Weibull distribution and the Rayleigh distribution, the results show that the Normal distribution, the Lognormal distribution and the Weibull distribution have the potential to describe the strength distribution of frozen clay. Due to space limitation, only these three distributions are considered and listed in the appendix.

3.1. Distribution laws of strength of warm frozen clay

Substituting each group of experimental strengths of warm frozen clay at different temperatures into Normal distribution, Lognormal distribution and Weibull distribution, the corresponding unknown parameters were obtained. Then, their theoretical probability values at distinct strength values were determined. Fig. 4 illustrates the experimental and theoretical probability values of warm frozen clay at −0.5 °C, −1.0 °C and −2.0 °C.

Fig. 4 a, b show that, when the temperatures of warm frozen clay are −0.5 °C and −1.0 °C, the theoretical values of the three probability distributions all approximate the experimental values. In order to analyze more intuitively the precision of the three kinds of probability distributions, the maximum difference values between experimental results and the theoretical distribution values are listed in Table 3. From Table 3, we can see that the maximum difference value between experimental results and the Weibull distribution is 0.09 and larger than those of the

<table>
<thead>
<tr>
<th>Temperatures</th>
<th>−0.5 °C</th>
<th>−1.0 °C</th>
<th>−2.0 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Warm frozen clay</td>
<td>0.747−0.947</td>
<td>0.819−1.633</td>
<td>1.286−1.918</td>
</tr>
<tr>
<td>Warm ice-rich frozen clay</td>
<td>0.551−0.824</td>
<td>0.795−1.173</td>
<td>1.213−1.655</td>
</tr>
</tbody>
</table>

Table 2

The variation ranges of strengths of two frozen clays at different temperatures (unit: MPa)
Fig. 4. Probability distributions of warm frozen clay at different temperatures.
other two distributions when temperature is $-0.5$ °C. When the temperature of warm frozen clay is $-1.0$ °C, the Lognormal distribution has a maximum difference between the experimental data and the theoretical distribution of 0.08 and is the largest of the three distributions. However, when the temperature of the frozen clay is $-2.0$ °C, the Normal and Lognormal distributions show large differences, both 0.17, between the experimental data and theoretical data (Fig. 4c). But the theoretical data for the Weibull distribution are close, and only 0.07 different, to the experimental data. Therefore, it is better to use the Weibull probability distribution to describe the strength of warm frozen clay.

3.2. Distribution laws of strength of warm ice-rich frozen clay

Similarly, every group of experimental strengths of the warm ice-rich frozen clay at different temperatures was fitted to determine for the unknown parameters. Their theoretical probability distributions were determined. Fig. 5 shows the experimental and theoretical probability distributions of the warm ice-rich frozen clay at the different temperatures ($-0.5$ °C, $-1.0$ °C and $-2.0$ °C).

For the warm ice-rich frozen clay, there is little difference between the experimental and theoretical values, and it is difficult to judge which distribution is more suitable to describe the experimental data. In order to analyze exactly the precision of the three kinds of probability distributions, the maximum difference between the experimental and theoretical data for the three distributions at $-0.5$ °C, $-1.0$ °C and $-2.0$ °C are presented in Table 4. From this table, it can be seen that at $-0.5$ °C, the maximum difference between the three theoretical distributions and experiment data are the same and their values are all 0.04. However, the Normal distribution has less difference than the others at $-1.0$ °C. When the temperature of warm ice-rich frozen clay is $-2.0$ °C, the Weibull distribution has less difference, 0.03, between the experimental data and theoretical data. In general, from Fig. 5 and Table 4, it can be concluded that the probability distribution law for the warm ice-rich frozen clay is satisfied slightly better by the Weibull distribution rather than the other distributions.

<table>
<thead>
<tr>
<th>Temperatures</th>
<th>$-0.5$ °C</th>
<th>$-1.0$ °C</th>
<th>$-2.0$ °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal distribution</td>
<td>0.06</td>
<td>0.06</td>
<td>0.17</td>
</tr>
<tr>
<td>Lognormal distribution</td>
<td>0.06</td>
<td>0.08</td>
<td>0.17</td>
</tr>
<tr>
<td>Weibull distribution</td>
<td>0.09</td>
<td>0.07</td>
<td>0.07</td>
</tr>
</tbody>
</table>

4. Stochastic damage constitutive model

Based on the general analyses of strength distributions, we can conclude that the Weibull distribution could well describe the strength distribution law for the warm and warm ice-rich frozen clay. Thus, a stochastic damage constitutive model was deduced using the above conclusions and theories of damage mechanics and probability, as well as statistic theory.

4.1. Damage constitutive relationship

According to the Lemaitre principle of equivalent stress (Lemaitre and Chaboche, 1970), the strain caused by $\sigma$ applied to a damaged material is equivalent to the strain caused by $\tilde{\sigma}$ acting on the undamaged material. Hence, the stress-strain relation for frozen soil under uniaxial condition can be expressed as follows:

$$\varepsilon = \frac{\sigma}{E} = \frac{\tilde{\sigma}}{E} = E(1 - D)$$

The above expression can be rewritten as:

$$\sigma = E\varepsilon(1 - D) \tag{2}$$

Where $E$ is the Young’s modulus of the damaged material. $E$ denotes the Young’s modulus of undamaged material. $D$ is a damage variable, and is defined as:

$$D = \frac{N_t}{N} \tag{3}$$

where $N_t$ is the number of damaged elements, and $N$ denotes all the elements.

4.2. Stochastic damage constitutive model based on the Weibull distribution

If the axial strain is regarded as a random variable, the probability density function of the Weibull distribution can be expressed as (Sheng et al., 1979):

$$f(x) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\eta}\right)^\beta\right], (x>0) \tag{4}$$

When the axial strain reaches the value of $\varepsilon$, the number of damaged elements, $N_t$, may be written as:

$$N_t(\varepsilon) = \int_0^\varepsilon Nf(x)dx \tag{5}$$

Then, substituting Eqs. (4) and (5) into Eq. (3), we have

$$D = \frac{N_t}{N} = 1 - \exp\left[-\left(\frac{\varepsilon}{\eta}\right)^\beta\right] \tag{6}$$

Substituting Eq. (6) into Eq. (2), the following formula can be obtained:

$$\sigma = E\varepsilon \exp\left[-\left(\frac{\varepsilon}{\eta}\right)^\beta\right] \tag{7}$$
Fig. 5. Probability distributions of warm ice-rich frozen clay at different temperatures.
Eq. (7), should satisfy the following conditions:

① when $\varepsilon = 0, \sigma = 0$
② when $\varepsilon = 0, \frac{d\sigma}{d\varepsilon} = E$
③ when $\varepsilon = \varepsilon_{\sigma_{\text{max}}}, \sigma = \sigma_{\text{max}}$
④ when $\varepsilon = \varepsilon_{\sigma_{\text{max}}}, \frac{d\sigma}{d\varepsilon} = 0$

where $\varepsilon_{\sigma_{\text{max}}}$ is strain at which the stress is equal to a peak stress $\sigma_{\text{max}}$.

Differentiating Eq. (7), we can obtain the following expression:

$$\frac{d\sigma}{d\varepsilon} = E \exp \left[ -\left( \frac{\varepsilon}{\eta} \right)^\beta \right] \left[ 1 - \beta \left( \frac{\varepsilon}{\eta} \right)^\beta \right]$$  (8)

 Obviously, Eqs. (7) and (8) satisfy the conditions of ① and ②.

If the conditions of ③ and ④ are substituted into Eqs. (7) and (8), the values of $\eta$ and $\beta$ can be obtained as follows:

$$\eta = \frac{\varepsilon_{\sigma_{\text{max}}}}{(\frac{1}{\beta})^\beta}$$  (9)

and

$$\beta = \frac{1}{\ln \frac{\varepsilon_{\sigma_{\text{max}}}}{\sigma_{\text{max}}}}$$  (10)

Eq. (10) is complex and if Eqs. (9) and (10) are synchronously substituted into Eq. (7), it is difficult to solve

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Table 4
Maximum differences between experimental and theoretical distributions for warm ice-rich frozen clay

<table>
<thead>
<tr>
<th>Temperatures</th>
<th>$-0.5 , ^{\circ}\text{C}$</th>
<th>$-1.0 , ^{\circ}\text{C}$</th>
<th>$-2.0 , ^{\circ}\text{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal distribution</td>
<td>0.04</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>Lognormal distribution</td>
<td>0.04</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>Weibull distribution</td>
<td>0.04</td>
<td>0.04</td>
<td>0.03</td>
</tr>
</tbody>
</table>

---

Fig. 6. Comparison of experimental and theoretical stress–strain curves for the warm frozen clay at $-0.5 \, ^{\circ}\text{C}$. 

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Eq. (7) for $E$ by the nonlinear regression method. Therefore, only Eq. (9) was used and the Eq. (7) will be changed to

$$
\sigma = E \varepsilon \exp \left[ -\frac{1}{\beta_i} \left( \frac{\varepsilon}{\varepsilon_{\max}} \right)^{\beta_i} \right]
$$

(11)

From Eqs. (10) and (11), it can be seen that the stress, $\sigma$, is determined by Young’s modulus of $E$ as well as the peak stress $\sigma_{\max}$ and the corresponding strain, $\varepsilon_{\max}$.

4.3. Verification of the stochastic damage constitutive model

The experimental results of the warm frozen clay and warm ice-rich frozen clay at different temperatures were substituted into Eq. (11), and $E_i$ and $\beta_i$ for each specimen were obtained by the nonlinear regressive method. The theoretically calculated stress-strain curves of stochastic damage constitutive model for every specimen can be drawn.

4.3.1. Discussion of the theoretical stress–strain curves for warm frozen clay

Because of space limitation, only a few of the experiments at the three testing temperatures are listed and discussed as follows.

From Figs. 6–8, it can be seen that the stochastic damage constitutive model based on the Weibull distribution can well describe the experimental results and reflect their deformation characteristics. In particular,

![Graph](image)

Fig. 7. Comparison of experimental and theoretical stress–strain curves for the warm frozen clay at $-1.0 \, ^\circ\text{C}$. 
when these specimens started to be damaged, the theoretical curves almost overlap with the experimental data. In order to evaluate the regressive effects, the experimental and theoretical peak stresses, $\sigma_{\text{max}}$, are also listed in Tables 5, 6, 7. The ratios of differences between experimental and theoretical average peak stress decrease with decreasing temperature, and they were 5.8%, 3.9%, 2.6% at $-0.5\, ^\circ\text{C}$, $-1.0\, ^\circ\text{C}$ and $-2.0\, ^\circ\text{C}$, respectively. That is, the lower the temperature of the warm frozen clay, the nearer the theoretical strength is to the experimental value. Similarly, as the temperature of the warm frozen clay

![Graph](image)

**Fig. 8.** Comparison of experimental and theoretical stress–strain curves for the warm frozen clay at $-2.0\, ^\circ\text{C}$.

<table>
<thead>
<tr>
<th>Table 5</th>
<th>Statistics on the average values for the warm frozen clay at $-0.5, ^\circ\text{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical indexes</td>
<td>E/MPa</td>
</tr>
<tr>
<td>Average value</td>
<td>43.78</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>7.38</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 6</th>
<th>Statistics on the average values for the warm frozen clay at $-1.0, ^\circ\text{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical indexes</td>
<td>E/MPa</td>
</tr>
<tr>
<td>Average value</td>
<td>57.19</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>9.99</td>
</tr>
</tbody>
</table>
decreases, its Young’s modulus increases. All of these phenomena may be caused by the fact that the bonding forces in the warm frozen clay became higher and higher with decrease in soil temperature.

4.3.2. Discussion on the theoretical stress–strain curves for the warm ice-rich frozen clay

Several typical experiments for the warm ice-rich frozen clay are shown in Figs. 9–11. From these figures, we find that the theoretical curves almost coincide with the experimental curves, which implies that the stochastic damage constitutive model deduced in this paper could well describe the whole stress–strain process of the warm ice-rich frozen clay under uniaxial condition. Similarly, the statistics of the average values of warm ice-rich frozen clay at the three different temperatures are also listed in Tables 8, 9, 10. Compared with the warm frozen clay, the ratios of differences for the experimental and theoretical mean peak stresses.

<table>
<thead>
<tr>
<th>Statistical indexes</th>
<th>$E$/MPa</th>
<th>$\beta$</th>
<th>$\varepsilon_{\sigma_{\max}}$</th>
<th>$\sigma_{\max}$/MPa</th>
<th>$\sigma_{\text{max}}$/MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average value</td>
<td>66.99</td>
<td>2.203</td>
<td>0.038</td>
<td>1.629</td>
<td>1.587</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>9.92</td>
<td>0.461</td>
<td>0.004</td>
<td>0.147</td>
<td>0.151</td>
</tr>
</tbody>
</table>

Fig. 9. Comparison of experimental and theoretical stress–strain curves for the warm ice-rich frozen clay at $-0.5 \degree C$. 

Table 7
Statistics on the average values for the warm frozen clay at $-2.0 \degree C$
σ_{max}, are slightly lower. Among these ratios, the maximum value is at −0.5 °C and its value is only 2.6%. Furthermore, because of the large quantity of ice in the warm ice-rich frozen clay, the Young’s modulus of warm ice-rich frozen clay is greater than that of warm frozen clay. However, the strength for warm ice-rich clay is smaller.

5. Probability analyses on strength

Due to the large randomness of strength of the warm frozen clay and warm ice-rich frozen clay, there exists an unsafe factor when the average strengths are used in engineering design (See Tables 11 and 12). From these two tables, it can be seen that if the average strengths for the warm frozen clay and warm ice-rich frozen clay are taken as design strength, there is large danger in practical engineering, and its failure probability will be 0.438–0.467. In order to ensure safety in engineering, letting the design strength be equal to a minimum experimental value will lead to greatly increased cost. However, this problem can be solved if the probability method is introduced into engineering designing. The confidence interval is worked out from the fact that the strength distribution laws for the warm frozen clay and warm ice-rich frozen clay satisfy the Weibull distribution.

The confidence levels for the warm frozen clay and warm ice-rich clay are listed in Tables 13 and 14, respectively. From Table 13, we can assess the safety of the warm frozen clay by the probability method. For
example, at −0.5 °C, when the uniaxial compressive strength, $\sigma_p$, is taken as 0.764 MPa, the safe reliability of warm frozen clay is 90%; and when $\sigma_p=0.735$ MPa, the undamaged probability is 95%. If the construction is very important, we can take the uniaxial compressive strength of warm frozen clay as 0.674 MPa, and its failure probability is only 1%. Similarly, the safe probability for the warm frozen clay at the other two temperatures could be determined by the above method. In the same way, from Table 14, we can obtain the safe reliability of warm ice-

![Fig. 11. Comparison of experimental and theoretical stress–strain curves for the warm ice-rich frozen clay at −2.0 °C.](image)

<table>
<thead>
<tr>
<th>Table 8</th>
<th>Statistics on the average values for warm ice-rich frozen clay at −0.5 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical indexes</td>
<td>$E$/MPa</td>
</tr>
<tr>
<td>Average value</td>
<td>849.71</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>248.48</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 9</th>
<th>Statistics on the average values for warm ice-rich frozen clay at −1.0 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistical indexes</td>
<td>$E$/MPa</td>
</tr>
<tr>
<td>Average value</td>
<td>1042.62</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>354.16</td>
</tr>
</tbody>
</table>


rich frozen clay at −0.5 °C, −1.0 °C and −2.0 °C, respectively.

6. Conclusions

Since there are many defects such as fissures and cavities in warm frozen clay and warm ice-rich frozen clay, their strengths are distributed randomly, as shown by a large numbers of experiments at different temperature (−0.5 °C, −1.0 °C and −2.0 °C). Based on these experimental data, it was found that the strengths of warm frozen clay and warm ice-rich frozen clay satisfy the Weibull distribution. A stochastic damage constitutive model for warm frozen clay and warm ice-rich frozen clay was developed by applying continuous damage theory and probability, as well as probability theory. The model was verified by a large amount of experimental data. From all these experiments and analyses, the following conclusions can be made.

(1) Through a series of experimental data, it was found that the stress–strain relationship of warm frozen clay shows a strain softening. For the warm ice-rich frozen clay, because there is much ice, its stress–strain relationship behaves in an elastic-plastic way instead of strain softening.

(2) The discreteness of the stress–strain curves for warm frozen clay becomes larger with increasing deformation. While for the warm ice-rich frozen clay, the discreteness of the stress–strain curves almost remains the unchanged even after yielding.

(3) Among several potential probability distributions, the Weibull distribution was found to best describe the strength law for warm frozen clay and warm ice-rich frozen clay.

(4) The stochastic damage constitutive model deduced in this paper could describe the stress–strain relationships for warm frozen clay, especially the warm ice-rich frozen clay well.

(5) The presented confidence levels of strength for warm frozen clay and warm ice-rich frozen clay are references for engineers in cold regions engineering.

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Appendix A. — Potential probability distributions

A.1. Normal distribution

As we know, many stochastic variables in production and scientific experiments can be described by the Normal distribution. For example, when the production conditions are the same, some indexes of production, such as compressive strength and length, satisfy the Normal distribution whose probability function \( F(x) \), is given by following:

\[
F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt
\]  

(1)
where $\mu$ and $\sigma (\sigma > 0)$ are parameters to be determined. If $x_1, x_2, \ldots, x_n$ are a group of experimental values, $\mu$ and $\sigma$ can be solved by the maximum likelihood method.

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$  \hspace{1cm} (a2)

A.2. Lognormal distribution

Ma et al. (2005) thought that the strength of concrete approximately satisfied the Lognormal distribution. So this distribution is assumed to describe compressive strength law for warm frozen clay and warm ice-rich frozen clay under uniaxial compression.

The probability function of the Lognormal distribution, $F(x)$, could be expressed as follows:

$$F(x) = \frac{1}{\sqrt{2\pi \sigma}} \int_{-\infty}^{x} \frac{1}{t} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} dt$$  \hspace{1cm} (a3)

$\mu$ and $\sigma$ can be obtained by the maximum likelihood method of continuous parent distribution if $x_1, x_2, \ldots, x_n$ are experimental values.

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (\ln x_i - \mu)^2$$  \hspace{1cm} (a4)

A.3. Weibull distribution

Wu (1995) and Cao et al. (1998) considered that the strengths of concrete and rock obey to the Weibull probability distribution. Here, this distribution is employed for the strength analysis of frozen soil.

The probability density function can be defined as follows:

$$f(x) = \frac{\beta}{\eta} \left( \frac{x}{\eta} \right)^{\beta-1} e^{-\left( \frac{x}{\eta} \right)^\beta}, \quad (x>0)$$  \hspace{1cm} (a5)

where, $\beta$ is a shape parameter, and $\eta$ denotes a scale parameter.

Similarly, $\beta$ and $\eta$ can be determined by maximum likelihood method using the following likelihood function:

$$L = \prod_{i=1}^{n} \left( \frac{\beta}{\eta} \left( \frac{x_i}{\eta} \right)^{\beta-1} e^{-\left( \frac{x_i}{\eta} \right)^\beta} \right)$$  \hspace{1cm} (a6)

Consequently,

$$\ln L = n \ln \left( \frac{\beta}{\eta} \right) + (\beta - 1) \sum_{i=1}^{n} \ln \left( \frac{x_i}{\eta} \right) - \sum_{i=1}^{n} \left( \frac{x_i}{\eta} \right)^\beta$$  \hspace{1cm} (a7)

Differentiating the function above for $\beta$ and $\eta$ and making their partial derivative function be equal to zero, we can obtain the following equations:

$$\begin{cases}
\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln \left( \frac{x_i}{\eta} \right) - \sum_{i=1}^{n} \left( \frac{x_i}{\eta} \right)^\beta \ln \left( \frac{x_i}{\eta} \right) = 0 \\
\frac{\partial \ln L}{\partial \eta} = -\frac{n}{\eta} + n(1 - \beta) + \frac{\beta}{\eta} \sum_{i=1}^{n} \left( \frac{x_i}{\eta} \right)^\beta = 0
\end{cases}$$  \hspace{1cm} (a8)

The proximate values of $\beta$ and $\eta$ can be obtained by using Newton’s method, and then the probability function can be given as following expression:

$$F(x) = \int_{-\infty}^{x} \frac{\beta}{\eta} \left( \frac{t}{\eta} \right)^{\beta-1} e^{-\left( \frac{t}{\eta} \right)^\beta} dt$$  \hspace{1cm} (a9)

References


