Mobilized shear strength of spatially variable soils under simple stress states

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**Abstract**

The spatial averaged shear strength is associated with a prescribed finite size spatial domain. It is not intended to cover the mobilized strength along a slip curve arising as a solution to a boundary value problem in a spatially variable medium. Nonetheless, the concept of strength in soil mechanics is fundamentally related to the mobilized strength along a slip curve. In this study, a plane strain soil specimen 12.8 m wide by 48 m high is subjected to undrained compression and shear via finite element analysis (FEA). The yield stress recorded before FEA fails to converge is defined as the mobilized strength. This mobilized strength is equal to the average strength. The statistical properties of the latter must depend to some extent on the physical problem, because a slip curve is not prescribed but is a solution of a boundary value problem in a spatially variable medium. One should expect the trajectory of the slip curve to be different for each realization of the random field. The statistical properties of the former are purely a function of the random field.
dominant role in determining the trajectory of the slip curve than the specific spatial variation in a realization. This implies that the broad commonsensical idea that a slip curve should “seek” weak spots in the spatially variable field may not even be the dominant mechanism.

The spatial average is mathematically well-defined. It is accurate to say that it is not intended to cover the mobilized strength along a slip curve arising as a solution to a boundary value problem in a spatially variable medium. Nonetheless, the concept of strength in soil mechanics is fundamentally related to the mobilized strength along a slip curve. In fact, strength parameters are typically evaluated by subjecting laboratory specimens (meso-scale) to prescribed stress boundaries. These strength parameters are applied to engineering scale (macro-scale) problems usually in the context of finite element analysis (FEA). The mobilized strength of meso-scale spatially variable specimens has been studied numerically [8–11]. However, there are no attempts to relate this meso-scale strength to the spatial average concept, which is theoretically useful and practically easy to evaluate. In particular, it is important to clarify the difference between averaging over a prescribed domain and averaging along an emergent slip curve and quantify the extent of the difference if possible.

The intent of this paper is to study the mobilized shear strength in a spatially variable medium subjected to simple stress states. A plane strain soil specimen 12.8 m wide by 48 m high is subjected in a spatially variable medium subjected to simple stress states. A two dimensional stationary random field for shear strength $\tau(x,z)$ can be characterized by its inherent mean value $E(\tau_i)$, inherent variance $\text{Var}(\tau_i)$, and auto-correlation function, in which $x =$ horizontal coordinate and $z =$ vertical coordinate. The auto-correlation function of a stationary random field $\tau_i(x,z)$ is defined to the correlation between two locations with $\Delta x$ apart horizontally and $\Delta z$ apart vertically:

$$\rho(\Delta x, \Delta z) = \rho(\tau_i(x,z), \tau_i(x+\Delta x, z+\Delta z))$$

$$= \frac{\text{Cov}((\tau_i(x,z), \tau_i(x+\Delta x, z+\Delta z)))}{\sqrt{\text{Var}(\tau_i(x,z))} \cdot \sqrt{\text{Var}(\tau_i(x+\Delta x, z+\Delta z))}}$$

\[ (1) \]

where $\text{Var}(\cdot)$ denotes variance; $\text{Cov}(\cdot)$ denotes covariance. The hypothesis of second-order stationarity allows $\rho$ to be function of $\Delta x$ and $\Delta z$ only, rather than the absolute coordinates $(x,z)$ and $(x+\Delta x, z+\Delta z)$ (see left hand side of Eq. (1)). The most popular auto-correlation model is the single exponential model:

$$\rho(\Delta x, \Delta z) = \exp \left(-\frac{2|\Delta x|}{\delta_x} - \frac{2|\Delta z|}{\delta_z} \right)$$

\[ (2) \]

where $\delta_x$ and $\delta_z$ are respectively the scales of fluctuation (SOF) in the $x$ and $z$ directions. It is clear that the correlation decreases as $\Delta x$ and $\Delta z$ increase. This is commonly observed in natural soils: soil properties are strongly correlated within a small interval and are weakly correlated when located far apart. The SOF is proportional to the correlation length, i.e. the length scale within which two locations are significantly correlated [2].

Vanmarcke [1] pointed out that the spatial average of soil properties over a region $D$ has mean value identical to the inherent mean but has variance less than the inherent one. Let the region $D$ be a rectangular domain defined by $[x_L, x_U][z_L, z_U]$. In other words, $(x_U, z_U)$ is the center of a rectangle of horizontal length $= L_x$ and vertical length $= L_z$. Mathematically, the spatial average over $D$ is defined as

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D variance reduction factor for the two-dimensional
\varepsilon \text{ safety factor}
\delta \text{ scale of fluctuation}
\delta_x \text{ horizontal scale of fluctuation}
\delta_z \text{ vertical scale of fluctuation}
UC \text{ unconfined compression}
\text{Var}(\cdot) \text{ variance}
\mu \text{ mean value}
\phi \text{ friction angle}
\text{covariance}
\text{rectangular domain}
E(\cdot) \text{ inherent (point) mean of the } \tau_i(x,z) \text{ random field}
E(\cdot) \text{ mean value}
FEA \text{ finite element analysis}
F \text{ vertical coordinate}
\text{cohesion}
\text{line average of } S
\text{line average of } \tau
\text{mobilized shear strength}
\text{line average of } \tau_s
\text{line average of } \tau_i
\text{the line average process of } \tau_{PA} \text{ (a one-dimensional random field)}
\text{the auto-correlation function}
\text{horizontal coordinate}
\text{vertical coordinate}
\text{coefficient of variation}
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Random field and spatial averaging

Spatial variabilities of soil properties are usually modeled by random fields [1]. Among random field models, second-order stationary (or statistically homogeneous) random fields are widely used due to their simplicity and possibly the only practical version that can be characterized from limited data [12]. For brevity, the term “second-order stationary” is referred to as “stationary” for the rest of this paper. A two dimensional stationary random field for shear strength $\tau(x,z)$ is identified from FEA and the line average is defined in the context of finite element analysis (FEA). The mobilized strength and based on this mechanism, to illuminate the differences between spatial average and mobilized strength. The spatial variability scenarios under which spatial average is approximately applicable are highlighted.

**Symbol list**

- $\phi$ friction angle
- $\rho(x)$ auto-correlation function
- $F^2$ variance reduction factor
- $\varepsilon^2$ variance reduction factor for the two-dimensional spatial averaging over region $D$
- $\tau_f$ shear strength
- $\tau_f(x,z)$ point process of the $\tau_f$ random field
- $\tau_i$ spatial average of the $\tau_f(x,z)$ random field over region $D$
- $\tau_m$ mobilized shear strength
- $\tau_i^A$ line average of $\tau_i$ along a potential slip curve
- $\tau_i^A(x,z)$ the line average process of $\tau_{PA}$ (a one-dimensional random field)
- $\tau_f^A(x,z)$ after applying the element-level averaging (element size = $L_z$
- $\tau_i^S$ line average of $\tau_i$ along the actual slip curve
- $\tau_f^S$ variance reduction factor for the one-dimensional line averaging over a potential slip curve
- $c$ cohesion
- COV coefficient of variation
- $CV(\cdot)$ covariance
- $D$ rectangular domain
- $E(\tau_i)$ inherent (point) mean of the $\tau_i(x,z)$ random field
- $E(\cdot)$ mean value
- $FEA$ finite element analysis
- $F_s$ width and height of the region $D$
- $L_s$ equivalent numbers of SOF in the $x$ direction
- $L_z$ equivalent numbers of SOF in the $z$ direction
- $PSO$ particle swarm optimization
- $SF$ safety factor
- $\delta$ scale of fluctuation
- $\delta_x$ horizontal scale of fluctuation
- $\delta_z$ vertical scale of fluctuation
- $UC$ unconfined compression
- $\text{Var}(\cdot)$ inherent (point) variance of the $\tau_i(x,z)$ random field
- $\text{Var}(\cdot)$ variance
- $x$ horizontal coordinate
- $z$ vertical coordinate
showed that the mobilized shear strength, denoted by \( s \), is the average over region \( D \). However, it is not clear if the mobilized shear strength is defined as the meso-scale shear strength measured in a plane-strain space. Finite element analyses (FEA) are conducted. The FEA model is a plane-strain 51.2 m \( \times \) 12.8 m rectangular area with 0.8 m \( \times \) 0.8 m FEA mesh grids, as shown in Fig. 1, making up 1024 plane-strain elements. The bottom boundary is roller, and the lower-left-most node is a hinge. The unit weights of all elements are zeros, the Young’s modulus is prescribed a high value (400 MN/m²), and the Poisson ratio is 0.3.

The spatially varying shear strength \( \tau \) at a point is simulated by stationary Gaussian random fields with inherent mean \( E(\tau) = 50 \) kN/m², inherent standard deviation \( \text{Var}(\tau) = 10 \) kN/m², and horizontal SOF = \( \delta_x \) and vertical SOF = \( \delta_z \). Adjacent to the top and bottom boundaries are two elastic zones with thicknesses = 1.6 m, as seen in Fig. 1. The purpose of these two elastic zones is to prevent the influence of the boundaries on the failure mechanism. The domain of interest \( D \) is therefore the 48 m \( \times \) 12.8 m rectangular area (width = 12.8 m, height = 48 m), and there are 960 elements in this domain. The local averaging subdivision algorithm developed by Fenton and Vanmarcke [13] may be adopted to assign the simulated \( \tau \) to each element. To do this, element-level averaging within each 0.8 m \( \times \) 0.8 m element is required. In this study, a Fourier-series-based method [14] is adopted for the local averaging within each element. To keep the problem simple, the shear strength \( \tau \) is assumed to be independent of confining pressure, i.e., \( \delta = 0 \).

The actual spatial average over domain \( D \) can be estimated as the average of the 960 assigned \( \tau \) values, denoted by \( \tau^{\text{D}} \). A single realization of random field will produce a sample value of \( \tau^{\text{D}} \). In this study, 100 realizations are generated for the following five chosen SOFs: \( \delta = 0.1 \) m, 1 m, 10 m, 100 m, and 1000 m. The case with \( \delta = 1000 \) m is close to a homogeneous case. The left plot in Fig. 2 shows 100 \( \tau^{\text{D}} \) sample values for a case with \( \delta_x = \delta_z = \delta \).

Since \( \tau^{\text{D}} \) is the average over the 960 elements, its statistical properties are the same as the actual spatial average over region \( D \) [Eq. (3)]. Hence, the mean value of \( \tau^{\text{D}} \) is the inherent mean 50 kN/m², and the variance of \( \tau^{\text{D}} \) is equal to the inherent variance \( 10^2 \) (kN/m²)² multiplied by the following variance reduction factor:

\[
\Gamma_{D2} = \left( \frac{[2 \cdot 12.8 / \delta] - 1 + \exp(-2 \cdot 12.8 / \delta)}{2 \cdot 12.8^2 / \delta^2} \right) \left( \frac{[2 \cdot 48 / \delta] - 1 + \exp(-2 \cdot 48 / \delta)}{2 \cdot 48^2 / \delta^2} \right)
\]

The upper right and lower right plots in Fig. 2 show the sample average and sample variance for the \( \tau^{\text{D}} \) samples, respectively. They are normalized by the mean and variance predicted within the context of spatial averaging (based on the equations in [1]). It is clear that the discrete spatial average \( \tau^{\text{D}} \) (a random variable) is close to continuous spatial average defined in Eq. (3) (a random variable) over the range of chosen SOFs. Because of the Gaussian

**Fig. 1.** Finite element model for the plane strain compression test.

**Fig. 2.** Samples of \( \tau^{\text{D}} \), their average values and variances under different SOFs, normalized with respect to the prediction for spatial averaging (based on the equations in [1]).
assumption, agreement in mean and variance also implies that both random variables are close in the broader sense of their cumulative distribution functions.

**Simulation of mobilized shear strength in finite element analysis (FEA)**

Although it is now clear that spatial average is suitable for modeling the discrete spatial average, \( \sigma_f \), it is not yet clear if the same theory can be related to the mobilized shear strength \( \tau_f \) in some way. In this study, the mobilized shear strength means the mesoscale shear strength produced by the soil mass in domain \( D \) under plane strain compression/shear. This mobilized shear strength \( \tau_f \) can be simulated by shearing the soil mass to failure, which can be carried out in FEA. Note that the micro-scale shear strength, \( \tau_s \), is spatially varying from one element to another. Two types of stress states are considered: compression and pure shear tests. For the compression test, the two lateral boundaries are free. A normal compression stress is exerted at the top boundary until FEA fails to converge. The stress–strain curve (see left plot in Fig. 3) is then plotted, and the yield axial stress applied on the top boundary \( \sigma_0 \) is identified. Note that the analysis in this paper is limited to the \( \Phi = 0^\circ \) condition, under which \( \tau_y = c = \) the radius of the Mohr circle at failure. As a result, \( \tau_y^2 = \) the radius of \( \sigma_0/2 \). This FEA simulation is similar to the unconfined compression (UC) test in laboratory, except that it is in a plane strain condition, rather than a triaxial condition. The UC test is adopted here because it is found that the initial confining pressure has no effect on the stress state (C: compression, S: pure shear), while the second letter indicates the random field type (I: isotropic random field) and (b) \( \delta_h = \infty \), \( \delta_z = \delta \) (anisotropic random field). Together with the two types of stress states (compression and pure shear), there are four scenarios: (a) Scenario CI – compression test with \( \delta_h = \delta_z \); (b) Scenario CA – compression test with \( \delta_h = \infty \); (c) Scenario SI – pure shear test with \( \delta_h = \delta_z \); and (d) Scenario SA – pure shear test with \( \delta_h = \infty \). The first letter indicates the stress state (C: compression, S: pure shear), while the second letter indicates the random field type (I: isotropic \( \delta_h = \delta_z \), A: anisotropic \( \delta_h = \infty \)). Note that with the single exponential model, the CI and SI cases are in fact not truly isotropic even if \( \delta_h = \delta_z \); the symbol “I” is just for convenience.

**Comparison between \( \tau_f^m \) and \( \tau_f^0 \)**

A single realization of a random field will produce a sample value of \( \tau_f^0 \). Following the same approach described in the previous section, 100 realizations are generated for each of the five chosen SOFs. For each realization of the random field, the sample value of the discrete spatial average \( \tau_f^m \) can also be easily calculated. Fig. 4 shows the comparison between ten samples of \( \tau_f^m \) and \( \tau_f^0 \) under various scenarios and \( \delta = 0.1 \) m, 1 m, 10 m, 100 m, and 1000 m. Note that \( \delta = \delta_h = \delta_z \) for scenarios CI and SI (isotropic) but \( \delta = \delta \) for scenarios CA and SA (anisotropic, \( \delta_h = \infty \)). It is clear from visual inspection that \( \tau_f^m \) is not the same as \( \tau_f^0 \) and in fact, it is always less than \( \tau_f^0 \). For \( \delta = 1000 \) m (nearly homogeneous case), \( \tau_f^m \) and \( \tau_f^0 \) are roughly equal. For \( \delta = 0.1 \) m, \( \tau_f^m \) and \( \tau_f^0 \) are also roughly equal except for the scenario SA.

**Table 1** presents the statistics of \( \tau_f^m \) and \( \tau_f^0 \) for \( \delta = 0.1 \) m, 1 m, 10 m, and 1000 m, including the mean values, coefficients of variation (COVs), and 10% quantiles. The findings are summarized as follows:

(a) \( \delta = 1000 \) m) The statistics (mean values, COVs, and 10% quantiles) of \( \tau_f^m \) and \( \tau_f^0 \) are in close agreement. This is sensible because for extremely large SOF, the shear strength \( \tau_y (x, z) \) is nearly homogeneous in \( D \).

(b) \( \delta = 0.1 \) m) The mean values of \( \tau_f^m \) and \( \tau_f^0 \) for scenarios CI, CA, and SI are both close to 50 kN/m\(^2\). This is also reasonable because for extremely small SOF, the averaging effect is strong, so the mobilized shear strength has small variability and has mean value very close to the inherent mean 50 kN/m\(^2\). Moreover, the COVs for \( \tau_f^m \) are larger than those for \( \tau_f^0 \) in these three scenarios. This phenomenon also appears in other scenarios and other SOFs. This peculiar observation will be discussed in a later section. For scenarios SA, \( \tau_f^m \) and \( \tau_f^0 \) have very different mean values. This is because for scenario SA, the potential slip curve is horizontal, so the averaging effect does not take place. This can be seen in Fig. 5. In addition to this, the critical slip curve will pass through the weakest layer, and the \( \tau_y \) value for this weakest layer is exactly \( \tau_f^0 \). It is then clear that this \( \tau_f^0 \) is significantly less than the spatial average \( \tau_f^0 \) in every realization and hence, likewise for the respective means.

(c) \( \delta = 10 \) m) The mean values for \( \tau_f^m \) are less than those for \( \tau_f^0 \) for all scenarios. In fact, the mean values of \( \tau_f^m \) and \( \tau_f^0 \) show the largest discrepancy when \( \delta = 10 \) m. This peculiar observation will also be discussed in a later section. Similar to the cases with \( \delta = 0.1 \) m, the COVs for \( \tau_f^m \) are larger than those for \( \tau_f^0 \) for all scenarios.

**Table 1** presents the results for sample size = 100. For larger sample sizes, Fig. 6 shows the statistics (mean, standard deviation,
and 10% quantile) of \( s_m \) and \( s_D \) under scenario CI for various sample size of \( s_m \) (sample size = 10 to 100). The sample statistics of \( s_m \) are plotted as solid lines, while the exact statistics of \( s_D \) are plotted as dashed lines. The discrepancy between the statistics of \( s_m \) and \( s_D \) is evident when the sample size is sufficiently large. In fact, for most of the plots in the figure, the discrepancy is already clear when sample size = 100, i.e., the statistics in Table 1 should be reliable.

In general, the mean value of \( s_m \) may be comparable or less than that of \( s_D \). One condition for the former to be true appears to be sufficient spatial averaging along the slip curve. However, the COV of \( s_m \) is always larger than the COV of \( s_D \), unless the soil mass is almost homogeneous (very large SOF). Consequently, assuming that \( s_m \) is identical to \( s_D \) may overestimate the mean value and underestimate the COV of \( s_m \). Note that the above two errors are both unconservative. The 10% quantiles listed in Table 1 show are plotted as solid lines, while the exact statistics of \( s_D \) are plotted as dashed lines. The discrepancy between the statistics of \( s_m \) and \( s_D \) is evident when the sample size is sufficiently large. In fact, for most of the plots in the figure, the discrepancy is already clear when sample size = 100, i.e., the statistics in Table 1 should be reliable.

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**Table 1**
Sample statistics of \( s_m \) and \( s_D \) for various scenarios and SOFs (sample size = 100).

<table>
<thead>
<tr>
<th>( d ) = 0.1 m</th>
<th>CI</th>
<th>CA</th>
<th>SI</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (kN/m²)</td>
<td>( s_m^D )</td>
<td>50.0</td>
<td>50.0</td>
<td>50.0</td>
</tr>
<tr>
<td></td>
<td>( s_m^m )</td>
<td>49.7</td>
<td>49.2</td>
<td>49.3</td>
</tr>
<tr>
<td>COV</td>
<td>( s_m^D )</td>
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<td>8.9e-3</td>
<td>8.8e-4</td>
</tr>
<tr>
<td></td>
<td>( s_m^m )</td>
<td>2.4e-3</td>
<td>1.5e-2</td>
<td>2.7e-3</td>
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<tr>
<td>10% quantile (kN/m²)</td>
<td>( s_m^D )</td>
<td>49.9</td>
<td>49.4</td>
<td>49.9</td>
</tr>
<tr>
<td></td>
<td>( s_m^m )</td>
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<td>48.3</td>
<td>49.1</td>
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<td>( d ) = 10 m</td>
<td>Mean (kN/m²)</td>
<td>49.7</td>
<td>50.3</td>
<td>49.7</td>
</tr>
<tr>
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<td>44.3</td>
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<td>COV</td>
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<td>7.7e-2</td>
<td>5.1e-2</td>
</tr>
<tr>
<td></td>
<td>( s_m^m )</td>
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<td>1.3e-1</td>
<td>1.1e-1</td>
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<tr>
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<td>( s_m^D )</td>
<td>46.4</td>
<td>45.4</td>
<td>46.4</td>
</tr>
<tr>
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<td>( s_m^m )</td>
<td>39.5</td>
<td>37.1</td>
<td>31.7</td>
</tr>
<tr>
<td>( d ) = 1000 m</td>
<td>Mean (kN/m²)</td>
<td>48.9</td>
<td>50.7</td>
<td>48.9</td>
</tr>
<tr>
<td></td>
<td>( s_m^D )</td>
<td>47.7</td>
<td>49.5</td>
<td>47.0</td>
</tr>
<tr>
<td>COV</td>
<td>( s_m^D )</td>
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<td>0.21</td>
<td>0.20</td>
</tr>
<tr>
<td></td>
<td>( s_m^m )</td>
<td>0.20</td>
<td>0.21</td>
<td>0.20</td>
</tr>
<tr>
<td>10% quantile (kN/m²)</td>
<td>( s_m^D )</td>
<td>36.5</td>
<td>37.3</td>
<td>36.5</td>
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<tr>
<td></td>
<td>( s_m^m )</td>
<td>35.3</td>
<td>36.2</td>
<td>34.7</td>
</tr>
</tbody>
</table>

Fig. 5. \( s_m^m \) and \( s_m^D \) for a special case for scenario SA. Horizontal critical slip curve will appear in the weakest horizontal layer.

Fig. 6. Comparison between the statistics (mean, standard deviation, and 10% quantile) of \( s_m^m \) and \( s_m^D \) under scenario CI for various sample size – upper three plots are for \( d = 0.1 \) m, middle three plots are for \( d = 10 \) m, and the lower three plots are for \( d = 1000 \) m.
the combined effects of mean values and COVs. The 10% quantile is indicative of the lower tail of the distribution and the differences manifested are important for reliability analysis.

Based on the above observations, it is clear that $s_D$ is indeed the $s_m$. However, the simulated mobilized shear strength $s_m$ is not the same as $s_D$. Hence, spatial averaging is generally not applicable to the prediction of the mean and variance of $s_m$. This is to be expected, although not well emphasized and well demonstrated in the literature. The spatial average as shown in Eq. (3) is purely geometric, based on a prescribed path or domain. The actual mesoscale strength of a soil specimen depends on the average strength along a critical slip curve that depends in part on the boundary conditions and the specific spatial variability in a realization. This critical principle is studied in the next section. In contrast to a prescribed path, this critical slip curve changes from realization to realization. Hence, Vanmarcke's theory of spatial average is never intended to cover the mobilized shear strength studied herein.

Some peculiar observations in Table 1 are not well understood. As noted above, it will be shown that the critical principle to understand these observations is: $s_m$ is very close to the line average along critical (actual) slip surface obtained in FEA rather than the spatial averaging over domain D. The validity of this principle is first demonstrated below.

**Comparison with line averaging along actual slip curve**

As mentioned earlier, FEA is conducted for each realization of the random field. At the loading step just prior to failure to converge, the stress states for all elements are recorded. An elaborate algorithm is then executed in this study to identify the slip curve where shear failure actually occurs. This algorithm employs the safety factor (SF) defined by Pham and Fredlund [15]. A global search algorithm based on the particle swarm optimization (PSO) [16] is adopted to find the curve with SF = 1, i.e., the actual slip curve in FEA.

The search starts from globally finding the straight line with the minimum SF by using PSO. This initial search for the optimal straight line (stage 1) only involves two degrees of freedom, as shown in Fig. 7. Then, one more degree of freedom is added to the middle of the optimal straight line (second plot from the left), and this line is taken to be the initial solution for stage 2 of PSO. The end result for the PSO in stage 2 is a two-segment curve with minimum SF. More degrees of freedom are again added to this optimal curve, and a new stage (stage 3) of PSO is carried out. This process is continued until there is one degree of freedom per mesh grid. The final optimal curve is found to have SF fairly close to 1, typically less than 1.01, and always coincides with the plastic zone predicted by the FEA, as shown in Fig. 8.

The same algorithm is executed in each realization of the random field after FEA. The resulting final optimal curve is then taken to be the critical (actual) slip curve for that realization. The elements that the critical slip curve passes through are identified, and the assigned $t_\gamma$ values of these elements are then averaged with weights proportional to the traversing lengths. This average value is therefore the “line average” along the actual slip curve obtained in FEA, denoted by $t_\gamma$. Fig. 9 shows the histograms of the absolute values for the inclination angles of the actual slip curves obtained in FEA. For scenarios CI and CA, the inclination angles are close to 45°, and for scenarios SI and SA, they are close to 0°. This indicates that the inclination angles are largely dominated by mechanics rather than by the spatial variability. However, the vertical position of the actual slip curve is fairly random, depending on the random field realization.

Fig. 10 shows the comparison between $t_\gamma$ and $t_\gamma$. It is clear that $t_\gamma$ is very close to $t_\gamma$ for all scenarios and all SOFs. There are cases where $t_\gamma$ is slightly less than $t_\gamma$. Although not shown in detail, the minor difference is correlated well with the asperity (or irregular-
The actual slip curve obtained in FEA. For the cases where \( s_f^2 < s_f^P \), the actual slip curves typically have larger asperities, and \( s_f^2 < s_f^P \) because \( s_f^P \) is the composition of \( s_f^2 \) and the dilation effect produced by the asperity. This asperity effect is clearly secondary for the examples studied. Note that this asperity effect is always zero for scenario SA. Hence, \( s_f = s_f^P \) for scenario SA.

The above observations support the following principle: the mobilized shear strength simulated by FEA (\( s_f \)) of the actual slip curve obtained in FEA. For the cases where \( s_f^2 \) average along the actual slip curve obtained in FEA (\( s_f^P \)), not the spatial averaging over the entire region D (\( s_f^D \)). With this principle in mind, most of the peculiar observations in Table 1 can now be explained:

(a) Why are the COVs for \( s_f^P \) larger than those for \( s_f^2 \) for all scenarios?

Let us consider the scenario SI as an example. The potential slip curves are nearly horizontal lines for this scenario (see Fig. 9). There are in fact numerous potential slip curves, but only one of them is the actual slip curve. Denote the line average along a potential slip curve by \( s_f^P(z) \):

\[
s_f^P(z) = \frac{1}{12.8} \int_0^{12.8} \tau_i(x,z)dx
\]

where \( z \) is the elevation for the potential slip curve. Note that the averaging distance is 12.8 m, which is the width of the soil specimen (Fig. 1). In fact, the line average process \( s_f^P(z) \) itself is a stationary one-dimensional (1-D) Gaussian random field defined on the interval between 0 m and 48 – 12.8 × tan(inclination angle of slip curve) m. Let the line average along the actual slip curve be denoted by \( s_f^2 \). Because the actual slip curve is the most critical curve, it is clear that \( s_f^2 \) is the minimum value of the line average process \( s_f^P(z) \), i.e., \( s_f^2 = \min_z s_f^P(z) \) over a finite interval containing fully formed slip curves (i.e., slip curves spanning left to right boundaries of the soil sample). For Fig. 1, this finite interval covering all fully formed potential slip curves is 48 – 12.8 × tan(inclination angle of slip curve). Based on Fig. 10, \( s_f^2 \) should be very close to the mobilized shear strength \( s_f^P \). In contrast, \( s_f^P \) is close to the average of the line average process \( s_f^P(z) \).

Because the variability of the minimum is larger than the mean, the COV of \( s_f^2 \) (or equivalently the COV of \( s_f^P \)) is larger than the COV of \( s_f^P \). The above discussion is obviously valid for scenarios SI and SA and is also valid for scenarios CI and CA, although the line average should be taken along a line with inclination angle = 45°.

(b) Why do the mean values of \( s_f^P \) and \( s_f^2 \) show the largest discrepancy when \( \delta = 10 \) m for scenarios CI, CA, and SI?

Let us again consider the scenario SI as an example. Note that \( s_f^P(z) \) is the result of a one-dimensional (1-D) line averaging along a horizontal potential slip curve. Hence, spatial averaging can be applied to find the mean value and variance of \( s_f^P(z) \). As a result, the line average process \( s_f^P(z) \) varies around the inherent mean with a very small COV. For the other extreme \( \delta_x = \delta_z = 1000 \) m, no line average occurs. In fact, \( s_f^P(z) = 50 \) kN/m². As a result, the line average process \( s_f^P(z) \) is always zero for scenario SA. Hence, \( s_f^2 = s_f^P \) for scenario SA.

For the extreme case \( \delta_x = \delta_z = 0.1 \) m, \( s_f^2 \) is very small. Although \( s_f^2 \) is the minimum value of the line average process \( s_f^P(z) \), \( s_f^2 \) cannot be very far from the inherent mean 50 kN/m² because the line average process \( s_f^P(z) \) varies around the inherent mean with a very small COV. For the other extreme \( \delta_x = \delta_z = 50 \) kN/m². As a result, the line average process \( s_f^P(z) \) varies around the inherent mean with a very small COV that is comparable to the inherent COV. Because \( s_f^2 \) is the minimum value of the line average process \( s_f^P(z) \), the mean value of \( s_f^2 \) can be noticeably less than the inherent mean.

Note that there is a tradeoff here. As SOF decreases, the SOF of the 1-D random field \( s_f^P(z) \) also decreases, so \( s_f^P(z) \) oscillates quickly around its mean. As the number of equivalent independent segments is proportional to the [interval length] / [SOF of \( s_f^P(z) \)], the probability of the minimum over this finite interval (a random variable) being less than a given constant is higher when the number of equivalent independent segments is large. This tends to decrease the mean value of \( s_f^2 \).

However, as SOF decreases, the 1-D averaging effect becomes stronger so that \( s_f^2 \) becomes smaller. This leads to a smaller COV of the 1-D random field \( s_f^P(z) \) and tends to increase the mean value of \( s_f^2 \). More specifically, the probability of the minimum over this finite interval (a random variable) being less than a given constant is lower when COV is small. The smallest mean value of \( s_f^2 \) (or \( s_f^P \)) occurs when \( \delta_x = \delta_z = 0 \). The length of the potential slip curve ( 12.8 m for the scenario SI) because further increasing SOF will not make \( s_f^2 \) significantly closer to 1. This critical SOF value has been noted by past researchers [4–6]. The above discussion is also valid for the other two scenarios CI and CA.

(c) Why do the mean values of \( s_f^P \) and \( s_f^2 \) deviate by a large margin for scenario SI even when \( \delta_x = 0.1 \) m?

The tradeoff mentioned in (b) does not take place in scenario SA because the potential slip curve is horizontal and there is zero spatial variability in the horizontal direction. The result is that \( s_f^2 = 1 \) regardless of SOF. Consequently, the COV of the 1-D random field \( s_f^P(z) \) is always equal to the inherent COV regardless of \( \delta_x \), but the SOF of \( s_f^P(z) \) decreases [hence
The mean values of \( \tau_{fm} \) for scenario SA with various \( \delta_z \):

<table>
<thead>
<tr>
<th>( \delta_z (m) )</th>
<th>Mean of ( \tau_{fm} ) (kN/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>42.1</td>
</tr>
<tr>
<td>1</td>
<td>32.3</td>
</tr>
<tr>
<td>10</td>
<td>32.2</td>
</tr>
<tr>
<td>100</td>
<td>42.0</td>
</tr>
<tr>
<td>1000</td>
<td>48.8</td>
</tr>
</tbody>
</table>

\( \tau_f^{m}(z) \) oscillates quickly around its mean] with decreasing \( \delta_z \). Therefore, the mean value of \( \tau_f^{m} \) (or \( \tau_f^{h} \)) always decreases with decreasing \( \delta_z \). This phenomenon only occurs in scenario SA.

In summary, the value of \( \tau_f^{m} \) can be largely explained using the mechanism underlying the determination of \( \tau_f^{m} \). The mechanism consists of taking line averaging along the potential slip curves and taking the minimum value among these averages, rather than taking spatial average of the entire region D. The line averaging along the potential slip curves is a helpful explanatory model. When this line averaging effect is strong, the COV in the 1-D random field \( \tau_f^{h}(z) \) then becomes small, and the deviation between \( \tau_f^{m} \) and \( \tau_f^{h} \) also becomes small. For scenario SA, this strong line averaging can never happen, and this is why scenario SA behaves so differently from scenarios CI, CA, and SI for small SOF.

There is another peculiar observation for scenario SA in Table 1 that appears to contradict (c) above: the mean value of \( \tau_f^{m} \) does not always decrease with decreasing SOF. For \( \delta_z > 10 \) m, \( \tau_f^{m} \) does decrease with decreasing SOF, but for \( \delta_z < 10 \) m, \( \tau_f^{m} \) increases with decreasing SOF. This peculiar observation cannot be easily explained by the principle that \( \tau_f^{h} \) is close to \( \tau_f^{m} \). This peculiar observation will be clarified in the next section.

**Micro scale view – issue of element-level averaging**

Recall the main principle revealed in the previous section: \( \tau_f^{m} \) is close to \( \tau_f^{m} \) rather than \( \tau_f^{h} \). Although this principle is drawn from the meso-scale analyses discussed in the previous sections, it suggests that at the micro scale, element-level spatial averaging over each 0.8 m \( \times \) 0.8 m element may not always be a reasonable way to obtain the \( \tau_f \) value of each element in FEA. One may argue that the size 0.8 m is quite small, so this spatial averaging in the element level should be adequate for \( \delta_z = 1 \) m, 10 m, 100 m, and 1000 m, although may not be adequate for \( \delta_z = 0.1 \) m. In this section, some numerical evidence will be furnished to show that it is not entirely so, especially for scenarios SA. An attempt is made to clarify the peculiar observation highlighted at the end of the previous section for scenario SA, namely the mean value of \( \tau_f^{m} \) simulated by FEA does not always decrease with decreasing SOF. It will be made clear that this peculiar observation in the micro scale is consistent with the explanations underlying the peculiar observations found in the meso scale analyses.

The issue of element-level averaging is first illustrated using scenario SA. Fig. 11 shows a typical realization of the \( \tau_f(x,z) \) random field for scenario SA with \( \delta_z = 1 \) m. The left plot in the figure shows the non-discretized realization before applying element-level averaging, and the second left plot shows the non-discretized profile of \( \tau_f^{h}(z) \) before applying element-level averaging. This non-discretized random field can be readily simulated by the frequency (Fourier-series) method [14]. The maximum resolution of the digitally simulated field is a function of the maximum cut-off frequency, which is taken to be an extremely large number so that the total energy leak is insignificant. In other words, the mathematical reality of a non-discretized field can be approximated quite accurately using the frequency method.

Note that \( \tau_f^{h}(z) \) is the line average parallel to the width of the soil specimen as defined in Eq. (7). For scenario SA, \( \tau_f^{h}(z) = \tau_f(x,0,z) \) as \( \tau_f(x,z) \) is a constant function of \( x \). The right two plots show the result after applying the element-level averaging, denoted by \( \tau_f^{m}(z) \), where the averaging window size in the vertical direction \( L_z \) (element size) is 0.1 m and 0.8 m:

\[
\tau_f^{m}(z) = \int_{-L_z/2}^{L_z/2} \tau_f^{m}\left(z + \tau\right) d\tau
\]

The left two plots represent the correct outcomes (in the sense of being mathematically consistent with the random field model), but FEA cannot map these plots as its input without some method of discretizing the random field realization. One method of discretization is given in Eq. (9). The resulting \( \tau_f^{m}(z) \) values for window size (element size) of 0.8 m shown in the rightmost plot of Fig. 11 are implemented in FEA.

For scenario SA, potential slip curves are horizontal lines. Hence, the critical slip curve obtained in FEA is horizontal and passes through the band (row of horizontal elements) with the lowest \( \tau_f^{h}(z) \) value. This band demarcated by two dashed lines in the rightmost plot of Fig. 11. The lowest \( \tau_f^{h}(z) \) value is found in this band and is equal to 30.8 kN/m². Note that such a slip curve is not unique: any horizontal slip curve passing through the band has the same \( \tau_f^{h} \) value. For this particular example in scenario SA, \( \tau_f^{h} = 30.8 \) kN/m² is not consistent with the mathematical reality (original non-discretized random field). The line average along the actual slip curve can be readily found by searching the global minimum in the non-discretized \( \tau_f^{h}(z) \) profile, without the need to invoke FEA. For the particular example shown in Fig. 11, the correct \( \tau_f^{m} \) is found to be roughly 15.6 kN/m² (see the second left plot in Fig. 11), which is significantly less than 30.8 kN/m². This is because for this example \( \delta_z = 1 \) m is small, so the detailed vertical variation within the element is lost after the element-level averaging over the 0.8 m interval. For scenario SA with a horizontal critical slip curve, it is clear that the micro scale element-level averaging issue may surface to affect the meso/macro scale behavior of the soil mass significantly.

As a result, \( \tau_f^{m} \) simulated by FEA is in general not the same as the actual \( \tau_f^{m} \). The former is the line average along the slip curve ob-
tained in FEA, and the latter is the line average along the actual slip curve. In fact, the term “slip curve obtained in FEA” is inaccurate because FEA cannot capture the displacement discontinuity implied by a slip curve. Failure occurs as a “slip band” in FEA; the band must obviously be wider than the mesh size. The issue that (FEA $t_f^s$) is clearly related to mesh size, as one can readily see that this issue is more serious when the element size is large. For the element size 0.8 m, the discrepancy between (actual $t_f^s$) and (FEA $t_f^s$) is 15.6 kN/m² and (FEA $t_f^s$) = 30.8 kN/m² is large, but as the element size decreases, the discrepancy decreases. Fig. 11 shows that when the element size decreases to 0.1 m [i.e., in total there are $(51.2/0.1) \times (12.8/0.1) = 65536$ elements ], (FEA $t_f^s$) becomes 19.2 kN/m², which is significantly closer to (actual $t_f^s$) = 15.6 kN/m². Clearly, mesh size has a major effect on the discrepancy.

This indicates that for scenario SA, the above discrepancy cannot be eliminated unless the ratio (element size)/$d_s$ is less than 0.04. This means for our previous results with element size = 0.8 m, the discrepancy cannot be eliminated unless $d_s$ > 20 m. This conclusion can now explain the peculiar observation in Table 2, where it is found that $t_f^m$ does not always decrease with decreasing SOF although it should. The $t_f^m$ values in Table 2 for $d_s = 0.1$ m, 1 m, and 10 m are inaccurate because the element-level averaging over the 0.8 m × 0.8 m area is too coarse for small SOFs. This problem is generally known, but the magnitude of the error for the scenario SA studied herein over fairly practical mesh sizes is rather surprising. This issue of mesh size (or the discrepancy between (actual $t_f^s$) and (FEA $t_f^s$)) not only happens in scenario SA but also in other scenarios. The impact of mesh size on mobilized shear strength is further discussed in [17].

**Concluding remarks**

1. For the purpose of this study, “mobilized” shear strength is defined as the meso-scale shear strength measured in a plane strain compression/shear test. The statistical behaviors of the mobilized shear strength, denoted by $t_f^m$, CANNOT be explained by spatial averaging. The mean value of $t_f^m$ is always less than that of the spatial average over domain D, $t_f^s$. The coefficient of variation (COV) of $t_f^m$ is always larger than that of $t_f^s$. Hence, it is unconservative to assume that $t_f^m$ is identical to $t_f^s$.

2. $t_f^m$ is found to be approximately equal to the line average along the actual slip curve, denoted by $t_f^s$. The minor difference between $t_f^m$ and $t_f^s$ is due to dilation under shear, arising from the asperity (roughness) of the actual slip curve. However, the asperity effect is secondary for the examples studied. As a result, the mechanism for $t_f^m$ is nearly the same as that for $t_f^s$. It is insightful to compare $t_f^s$ with spatial average, $t_f^p$. $t_f^p$ is defined over a prescribed domain. It is purely a function of the random field. $t_f^p$ is the minimum among the line averages along all potential slip curves. The locations of the actual slip surfaces primarily depend on the stress boundaries with some perturbations from the specific spatial variation in a realization. Hence, the critical slip curve is not known prior to FEA. Due to the action of taking minimum, $t_f^p$ (or $t_f^m$) may have mean value less than that for $t_f^s$, especially when the line averaging effect along the potential slip curve is weak, and may have COV larger than that for $t_f^s$.

3. The element-level averaging method to assign a single value of the random field to each finite element may be problematic. This random field discretization at element-scale (called micro-scale) can affect the meso or larger scale outcomes in FEA, which are of engineering interest. The meso-scale $t_f^s$ simulated by FEA using element-level averaging may have mean value larger than the actual mean. These unconservative errors increase with mesh size as to be expected. These micro-scale discretization errors and the meso-scale errors noted in (1) share a common root. The issue of element-level averaging can be mitigated if the element size is made small. However, in the worst case where the potential slip curve is parallel to the direction of homogeneity in scenario SA ($d_s$ very large), it is found that the element size in the vertical direction needs to be smaller than $d_s \times 0.04$ to achieve an error less than 20% in an example.

4. The scenario where the stress state is pure shear and the random field is homogeneous in horizontal direction is the worst scenario for both meso and micro scales, in the sense that the $t_f^s$ simulated by FEA deviates the most from the spatial averaging $t_f^p$ in the meso scale and, at the same time, the mesh size issue is most problematic in the micro scale. Unfortunately, this scenario is not uncommon in geotechnical engineering, e.g., an embankment failure in a ground with layered soils. The bottom part of the failure surface for the embankment failure is typically in a stress state closer to pure shear.

5. The observations drawn from this study should apply to cases with simple and uniform stress states. However, realistic geotechnical problems are mostly subjected to non-uniform stress states, e.g., shallow foundation, embankment, slopes, excavations, etc. Research is ongoing to ascertain the validity of these observations to more realistic and more complex stress states.

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**References**
