Multiple-Input Multiple-Output (MIMO) radar with colocated antennas has an increased target parameter estimation performance, but at the cost of increased computational complexity. This paper first presents the conditions required for the interference covariance matrix (ICM) of colocated MIMO radar to take a special structure, namely a Kronecker product of some sub-ICMs, and then proves that based on this ICM structure, the conventional Minimum Variance Distortionless Response (MVDR) algorithm can be reformulated into a combination of three estimation algorithms all of much smaller scales, such that the computational complexity is decreased significantly. However, this ICM structure can be destroyed by inactive scattering sources, whose influence is studied via numerical experiments. It is found that inactive point scatterers can still be suppressed by adaptive algorithms relying on the ICM structure, on condition that the number is fewer than that of receiving antennas.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

According to the spacing of antennas, Multiple-Input Multiple-Output (MIMO) radar [1–3] is generally categorized into two types, MIMO radar with colocated antennas [4–7] and that with widely separated antennas [8–12]. Colocated MIMO radar can simultaneously transmit multiple different waveforms [13] and has more degrees of freedom [7], a higher target parameter identifiability [5] and a higher angular resolution [3]. Compared with conventional phased-array radar, colocated MIMO radar has an extra transmitting dimension to process in the receiving end, which greatly increases the scale of the adaptive estimation problem. If conventional adaptive estimation algorithms, such as the Minimum Variance Distortionless Response (MVDR) [14], are directly generalized to colocated MIMO radar, the computational complexity would be high and many training samples would be required to achieve an acceptable performance. Therefore, how to decrease the algorithm computational complexity and sample number requirement is an interesting topic for the research on colocated MIMO radar.

For the MIMO Space–Time Adaptive Processing (STAP) problem, using a fact that the clutter subspace has a low rank, the clutter space is represented by prolate spheroidal wave functions in [15,16], and then the computational complexity and sample number requirement are reduced, to a moderate extent however. For the MIMO beamforming problem, some algorithms, such as the MIMO Capon algorithm and the MIMO Amplitude-Phase Estimation (APES) algorithm proposed in [6], formulate the originally high dimensional adaptive algorithm into a combination of two algorithms both of much lower dimensions [3,6], such that the computational complexity and sample number requirement can be significantly reduced. The achievement depends highly on a special structure of the Interference Covariance Matrix (ICM), namely a Kronecker product of two sub-ICMs. The Two-Side Beamforming (TSB) algorithm...
algorithm relying on the ICM structure. Section 4 analyzes
the ICM structures of typical interference sources, and
discusses methods to select training samples from signals
in receiving channels to estimate unknown sub-ICMs.
Section 5 studies the impacts of inactive point scatterers
on algorithms relying on the Kronecker-product structure
of the ICM. Finally, Section 6 concludes the work.

2. Signal model for colocated MIMO radar with Doppler

Let us consider a colocated MIMO radar system with \( N \)
receiving antennas, and \( M \) transmitting antennas from
which \( M \) polyphase coded waveforms, denoted by
\( \mathbf{s}_1, \ldots, \mathbf{s}_M \in \mathbb{C}^{t 	imes 1} \), can be simultaneously illuminated, where \( L \)
denotes the number of subpulses of each waveform. Let us
define \( \mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_M]^T \in \mathbb{C}^{M \times L} \) as the matrix of transmitted
signals, where \( (\cdot)^T \) denotes the transpose. The signal
covariance matrix \( \mathbf{R}_s = \mathbf{S} \mathbf{S}^H / L \) is assumed to be full rank and thus
reversible, where \( (\cdot)^H \) denotes the conjugate transpose. If
\( \mathbf{R}_s = \mathbf{I} \), the transmitted beampattern is omnidirectional [13].

Let us assume that a generic target is present in the radar far
field, at a spatial location denoted by \( \theta \). The temporal
waveform transmitted into this spatial location can be
written as \( \mathbf{b}^T (\theta) \mathbf{S} \), where \( \mathbf{b}(\theta) \in \mathbb{C}^{M \times 1} \) denotes the
transmitting steering vector with respect to the spatial position \( \theta \).
The receiving steering vector with respect to \( \theta \) is denoted by
\( \mathbf{a}(\theta) \in \mathbb{C}^{N \times 1} \). Assume that the MIMO radar successively
illuminates and receives \( K \) snapshots and during \( K \) snap-
shots, the target does not move across a range bin. At the
\( k \)th snapshot, received signals with respect to the spatial
position \( \theta \) can be expressed by

\[
\mathbf{x}(k) = \mathbf{a}(\theta) \mathbf{b}(\theta)^T (\mathbf{S} + \mathbf{Z}_k), \quad k = 1, \ldots, K, \tag{1}
\]

where \( \mathbf{b}(\theta) \) denotes the target complex amplitude, and
\( \mathbf{Z}_k \in \mathbb{C}^{N \times 1} \) denotes background interferences. In what fol-

dows, \( \mathbf{a}(\theta) \) will be represented by \( \mathbf{a} \) for short, and \( \mathbf{b}(\theta) \) by \( \mathbf{b} \).

In practice, range compression is generally indispensable and
we will study range compressed signals subsequently.
Using the matrix \( \mathbf{SR}^\dagger / L \) for range compression, we can
obtain \( M \) channels associated with \( M \) transmitted waveforms,
where \( (\cdot)^\dagger = [(\cdot)^H]^{-1} \). The output signals can be written in a
matrix form as

\[
\mathbf{y}(k) = \mathbf{X}(k) \mathbf{S}^\dagger \mathbf{R}_s^{-1} / L = \mathbf{a}(\theta) \mathbf{b}^T + \mathbf{z}(k), \tag{2}
\]

where

\[
\mathbf{z}(k) = \mathbf{Z}(k) \mathbf{S}^\dagger \mathbf{R}_s^{-1} / L. \tag{3}
\]

Rewriting \( \mathbf{y}(k) \) into a vector form gives

\[
\mathbf{y}(k) = \text{vec}[\mathbf{Y}(k)] = \text{vec}[\mathbf{X}(k) \mathbf{S}^\dagger \mathbf{R}_s^{-1} / L] = \mathbf{b}(\theta) \otimes \mathbf{a} + \mathbf{z}(k), \tag{4}
\]

where \( \text{vec}(\cdot) \) denotes the vectorization operator, \( \otimes \) is the
Kronecker-product operator,

\[
\mathbf{z}(k) = \text{vec}[\mathbf{Z}(k)] = \left( \mathbf{R}_s^{-1} \mathbf{S}^\dagger / L \mathbf{I}_N \right) \text{vec}[\mathbf{Z}(k)]. \tag{5}
\]

(\cdot)^* \) denotes the conjugate operator, and \( \mathbf{I}_N \) denotes the
\( N \times N \) identity matrix. For signals received in \( K \) snapshots,
we use the following denotations: \( \mathbf{Y} = [\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(K)] \),
\( \mathbf{Z} = [\mathbf{z}(1), \mathbf{z}(2), \ldots, \mathbf{z}(K)]^\dagger \) and \( \mathbf{y} = \text{vec}(\mathbf{Y}) \). Assume that
during the \( K \) snapshots, the radar illuminates the same aspect
of the target and thus the target backscattering power can be considered to be invariant [11,12]. Under this

The rest of this paper is organized as follows: Section 2
shows the full-scale signal model for colocated MIMO
radar. Section 3 lists the assumptions required for back-
ground interferences to take a Kronecker-product struc-
tured ICM, and presents the signal processing scheme for
colocated MIMO radar based on an adaptive estimation

This ICM structure is a result of a critical assumption
[6] that received signals in different range bins before
range compression are statistically independent. How-
ever, such an assumption is generally satisfied in practice
for signals after range compression [17]. Before range
compression, returns of inactive scattering sources may
spread over several adjacent range bins. Therefore,
whether this assumption is still satisfied in this situation
deserves a deeper research. If some kinds of interferences
fail to satisfy this assumption, another problem of interest
is how the algorithms relying on this ICM structure would
perform in presence of those interferences.

In this paper, we concern the adaptive target param-
eter estimation problem in colocated MIMO radar. For
the full-scale MIMO signal model [15], we first show that
under some assumptions, the ICM can be written as a
Kronecker product of three sub-ICMs. Based on this ICM
structure, the MVDR algorithm is generalized into colo-
cated MIMO radar and then reformulated to be a combi-
nation of three adaptive estimation algorithms all of
much lower dimensions. This result gives rise to a signal
processing scheme for colocated MIMO radar that just
estimates unknown sub-ICMs and then inverses them.
Consequently, both the sample number requirement and
the computational complexity of this scheme are low.

Furthermore, the ICM structure of typical radar inter-
ferences is studied. It is found that if just white active
jamming and white channel thermal noise are present in
received signals, the ICM can be expressed by a Kronecker
product of three sub-ICMs, with two known and one
unknown. But when inactive scattering sources, such as
clutter and inactive point scatterers, are present, the
ICM does not take this structure anymore. Therefore, in
numerical experiments, via the MIMO beamforming prob-
lem, we study the impact of inactive point scatterers on
algorithms relying on the Kronecker-product structure
of the ICM. It is found that although inactive scattering
sources would destroy that ICM structure, algorithms
relying on the ICM structure can still reach the optimal
performance, on condition that the total number of
jamming sources plus inactive point scatterers is smaller
than that of receiving antennas. Once more interference
sources from different directions are present, algorithms
relying on the ICM structure does not converge to the
optimal performance anymore. In other words, although
inactive scattering sources may be present in received
signals, colocated MIMO radar can still use an adaptive
signal processing algorithm relying on the ICM structure,
in order to reap the benefits resulted, but the number of
receiving antennas should be greater than that of active
jamming sources plus inactive point scatterers.

The rest of this paper is organized as follows: Section 2
shows the full-scale signal model for colocated MIMO
radar. Section 3 lists the assumptions required for back-
ground interferences to take a Kronecker-product struc-
tured ICM, and presents the signal processing scheme for
colocated MIMO radar based on an adaptive estimation

proposed in [6] does not rely on this ICM structure, but
with a minor modification, it can be reformulated into a
new algorithm of a very close performance but relying on
this ICM structure.

Consequently, both the sample number requirement and
the computational complexity of this scheme are low.

Further, the ICM structure of typical radar inter-
ferences is studied. It is found that if just white active
jamming and white channel thermal noise are present in
received signals, the ICM can be expressed by a Kronecker
product of three sub-ICMs, with two known and one
unknown. But when inactive scattering sources, such as
clutter and inactive point scatterers, are present, the
ICM does not take this structure anymore. Therefore, in
numerical experiments, via the MIMO beamforming prob-
lem, we study the impact of inactive point scatterers on
algorithms relying on the Kronecker-product structure
of the ICM. It is found that although inactive scattering
sources would destroy that ICM structure, algorithms
relying on the ICM structure can still reach the optimal
performance, on condition that the total number of
jamming sources plus inactive point scatterers is smaller
than that of receiving antennas. Once more interference
sources from different directions are present, algorithms
relying on the ICM structure does not converge to the
optimal performance anymore. In other words, although
inactive scattering sources may be present in received
signals, colocated MIMO radar can still use an adaptive
signal processing algorithm relying on the ICM structure,
in order to reap the benefits resulted, but the number of
receiving antennas should be greater than that of active
jamming sources plus inactive point scatterers.
assumption, the vector $\beta = (\beta(1), \ldots, \beta(K))^T$ can be expressed by a common target amplitude $\beta$ multiplied by a Doppler steering vector denoted by $c \in \mathbb{C}^{K \times 1}$. With above denotations, we can write the full-scale MIMO signal model as

$$y = \beta d + z,$$  

(6)

where $d = c \otimes b \otimes a \in \mathbb{C}^{MNK \times 1}$ denotes the target signal signature of interest. Similar to the MIMO STAP signal model concerned in [15,16], this signal model takes multiple snapshots into consideration. Different from the full-scale signal model, the MIMO beamforming signal model [3,6,18] just considers signals received at one snapshot. Similar to the MIMO STAP signal model, the MIMO beamforming signal model [3,6,18] just considers signals received at one snapshot. Different from the MIMO STAP problem [15,16], we focus on whether the Kronecker-product structure of the ICM can be used in adaptive parameter estimation algorithms for colocated MIMO radar to reduce the computational complexity and training sample number requirement.

3. Adaptive estimation with Kronecker-product structured ICM

From (6), we can find that the signal dimension for colocated MIMO radar increases significantly to MNK, compared with a phased-array radar with the same number of receiving antennas. In this section, we first generalize the classical MVDR algorithm [14] directly to the colocated MIMO radar case. Then we list the assumptions required for the ICM to take a Kronecker-product structure. Based on this ICM structure, the MIMO MVDR algorithm is then reformulated into a combination of three adaptive algorithms all of much lower scales. As a result, both the computational complexity and the required number of training samples decrease significantly.

3.1. Target amplitude estimation algorithm

The ICM in (6) can be written as $Q = E(z z^H)$, where $E(\cdot)$ denotes the expectation operator. The MVDR algorithm [14] tries to find a weighting vector that can minimize the variance of output signal while keeping the target response distortionless. For the signal model (6), this concept can be expressed by the following optimization problem [14]:

$$\min_w w^H Q w \quad \text{s.t.} \quad w^H d = 1.$$  

(7)

The solution is well-known as

$$w_{\text{MVDR}} = \frac{Q^{-1} d}{d^H Q^{-1} d}.$$  

(8)

With this weighting vector, we can obtain an estimate of the target amplitude as follows:

$$\hat{\beta}_{\text{est}} = \frac{d^H Q^{-1} y}{d^H Q^{-1} d}.$$  

(9)

In practice, the ICM $Q$ is generally unknown and should be estimated by using training samples. Denote by $Q$ as such an estimate. Substituting $Q$ for $Q$ in (8), we can obtain an adaptive weighting vector

$$w_{\text{adp}} = \frac{Q^{-1} d}{d^H Q^{-1} d}.$$  

(10)

In order to keep the signal to interference plus noise ratio (SINR) loss less than 3 dB, the number of training samples used to estimate $Q$ should be more than $2MNK$ [19]. In practice, however, even if an ideal estimate of $Q$ is available, the computational complexity to inverse an $MNK \times MNK$ matrix $Q$ is as high as $O(M^2 N^2 K^3)$ [20]. In [15,16], the partially adaptive technique is used to reduce the training sample number requirement and computational complexity, but the cost is still very high. The MIMO Capon algorithm [6] can reduce the algorithm complexity significantly, and the key point is the unintentionally used Kronecker-product structure of the ICM. However, the conditions required for an ICM to hold such a structure deserve a deeper research.

3.2. Conditions for ICM to hold the Kronecker-product structure

In what follows, we would list and briefly explain necessary assumptions for the ICM to hold a Kronecker-product structure.

1. The first assumption is that interference signals are zero-mean complex Gaussian distributed. Gaussian distribution is widely used to model radar interference signals. With this assumption, the statistical property of interference signals can be described by their covariance matrix. For convenience, let $z(k,l)$ represent the $i$th column of $Z(k)$. Then the cross covariance matrix between $z(k,i)$ and $z(l,j)$ can be written as

$$V_{ij}^u = E[z(k,i)z(l,j)^H], \quad k,l,i,j = 1, \ldots, K.$$  

(11)

From (5), the cross covariance matrix between $z(k)$ and $z(l)$ can be written as

$$M_{kl} = E[z(k)z(l)^H] = (R_z^{-1} S^*/L \otimes I_N) \times \begin{bmatrix} V_{11}^u & \cdots & V_{1K}^u \\ V_{21}^u & \cdots & V_{2K}^u \\ \vdots & \ddots & \vdots \\ V_{K1}^u & \cdots & V_{KK}^u \end{bmatrix} (S^* R_z^*/L \otimes I_N).$$  

(12)

2. The second assumption is that interference signals in different range bins before range compression are statistically independent. According to this assumption, $V_{ij}^u = \delta_{ij} V_{ii}^u$, where $\delta_{ij} = 1$ if $i=j$, and $\delta_{ij} = 0$ otherwise. Denote $V_{ii}^u = V_{ii}^u$ for short. In this case, the mid matrix in the right hand of Eq. (12) becomes a block diagonal matrix and then

$$M_{kl} = (R_z^{-1} S^*/L \otimes I_N) \begin{bmatrix} V_{ii}^u & 0 & 0 \\ 0 & V_{ii}^u & 0 \\ 0 & 0 & V_{ii}^u \end{bmatrix} (S^* R_z^*/L \otimes I_N).$$  

(14)

This assumption is critical for [6] to derive the MIMO Capon beamforming algorithm. In a physical point of view, for active jamming and noise, it means that their
bandwidths are wider than the radar sampling frequency. For inactive scattering sources, such as ground and cloud clutter, it means that sampled echoed signals before range compression are statistically independent. However, this assumption is generally made for signals after range compression [17]. Considering that echoed signals of an inactive scattering source may spread over several sampling points, the second assumption is vulnerable to inactive scattering sources. This assumption is indispensable to obtain a Kronecker-product structure. The Kronecker product association with this assumption, we have the following denotation:

\[ \mathbf{V}_k = \mathbf{V}_{kk} = \mathbf{V}_{lk}^1 = \mathbf{V}_{kk}^2 = \cdots. \]  

Based on this assumption, we can rewrite (14) as

\[ \mathbf{M}_k = \mathbf{M}_{lk} = (\mathbf{R}_s^{-T}\mathbf{S}^*/(L \otimes \mathbf{I}_N)) \times \begin{bmatrix} \mathbf{V}_k & 0 & 0 \\ 0 & \mathbf{V}_k & 0 \\ 0 & 0 & \mathbf{V}_k \end{bmatrix} \left( \mathbf{S}^{-T} \mathbf{R}_s^{-T}/(L \otimes \mathbf{I}_N) \right) = \mathbf{R}_s^{-T}/(L \otimes \mathbf{V}_k). \]  

From (16), the ICM of interferences at a snapshot takes the Kronecker-product structure. The Kronecker product has a useful property that the inverse matrix of a Kronecker product of two matrices is equal to the Kronecker product of their inverse matrices [20]. With this property, the inverse matrix of \( \mathbf{M}_k \) can be obtained by

\[ \mathbf{M}_k^{-1} = \mathbf{I} \mathbf{R}_s^{-T} \otimes \mathbf{V}_k^{-1}. \]  

4. The fourth assumption is that in each range bin, interference signals received at different snapshots correlate at a certain speed and the decorrelation speed is the same for interferences received by all antennas. According to this assumption, the cross covariance matrix between interference signals received at the \( k \)th and the \( l \)th snapshot, denoted by \( \mathbf{V}_{kl} \), obeys the following relationship:

\[ \mathbf{V}_{kl} = \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{V}_k^l = \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{V}_k. \]  

where \( \frac{\zeta_{kl}}{\varsigma_{kl}} \) denotes the collective correlation coefficient between interferences received at the two snapshots. If \( k = l, \frac{\zeta_{kl}}{\varsigma_{kl}} = 1 \). From (18), the cross covariance matrix \( \mathbf{M}_{kl} \) can be written as

\[ \mathbf{M}_{kl} = (\mathbf{R}_s^{-T}\mathbf{S}^*/(L \otimes \mathbf{I}_N)) \times \begin{bmatrix} \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{V}_k^l & 0 & 0 \\ 0 & \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{V}_k^l & 0 \\ 0 & 0 & \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{V}_k^l \end{bmatrix} \left( \mathbf{S}^{-T} \mathbf{R}_s^{-T}/(L \otimes \mathbf{I}_N) \right). \]  

5. To express \( \mathbf{M}_{kl} \) in a more concise form, the fifth assumption is that the decorrelation speeds for all range bins are the same, which can be expressed by

\[ \frac{\zeta_{kl}}{\varsigma_{kl}} = \frac{\zeta_{kl}}{\varsigma_{kl}} = \cdots = \frac{\zeta_{kl}}{\varsigma_{kl}}. \]  

In association with the third assumption, we can rewrite the cross covariance matrix \( \mathbf{M}_{kl} \) by

\[ \mathbf{M}_{kl} = \frac{\zeta_{kl}}{\varsigma_{kl}} (\mathbf{R}_s^{-T}\mathbf{S}^*/L \otimes \mathbf{I}_N) \begin{bmatrix} \mathbf{V}_k & 0 & 0 \\ 0 & \mathbf{V}_k & 0 \\ 0 & 0 & \mathbf{V}_k \end{bmatrix} \left( \mathbf{S}^{-T} \mathbf{R}_s^{-T}/L \otimes \mathbf{I}_N \right) = \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{R}_s^{-T} \otimes \mathbf{V}_k/L. \]  

6. The sixth assumption is that the statistical property of interference signals remains the same during \( K \) snapshots, which can be considered as the time-invariant property of interference signals. With this assumption, we have

\[ \mathbf{V} = \mathbf{V}_1 = \mathbf{V}_2 = \cdots = \mathbf{V}_K. \]  

From (22), the cross covariance matrix \( \mathbf{M}_{kl} \) can be formulated to

\[ \mathbf{M}_{kl} = \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{R}_s^{-T} \otimes \mathbf{V}/L = \frac{\zeta_{kl}}{\varsigma_{kl}} \mathbf{M}. \]  

With all the above assumptions, we can finally write the ICM \( \mathbf{Q} \) as

\[ \mathbf{Q} = \mathbf{E} (\mathbf{z} \mathbf{z}^H) = \mathbf{E} \begin{bmatrix} \mathbf{z}(1) \mathbf{z}(1)^H & \cdots & \mathbf{z}(1) \mathbf{z}(K)^H \\ \vdots & \ddots & \vdots \\ \mathbf{z}(K) \mathbf{z}(1)^H & \cdots & \mathbf{z}(K) \mathbf{z}(K)^H \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{11} & \cdots & \mathbf{M}_{1K} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{K1} & \cdots & \mathbf{M}_{KK} \end{bmatrix} = \mathbf{P} \otimes \mathbf{M} = \mathbf{P} \otimes \mathbf{R}_s^{-T}/L \otimes \mathbf{V}. \]  

where \( \mathbf{P} = \left( \frac{\zeta_{kl}}{\varsigma_{kl}} \right)_{K \times K} \) is termed as the decorrelation coefficient matrix here. From (24), under certain assumptions, the ICM \( \mathbf{Q} \) can indeed be written as a Kronecker product of three sub-ICMs. From (8), we need the inverse matrix of \( \mathbf{Q} \) to obtain the adaptive weighting vector. As is well known, inversing a matrix is a time-consuming operation. But according to the aforementioned property of the Kronecker product, the inverse matrix of \( \mathbf{Q} \) can be computed in a time-saving manner by

\[ \mathbf{Q}^{-1} = \mathbf{P}^{-1} \otimes \mathbf{R}_s^{-T} \otimes \mathbf{V}^{-1}. \]  

Therefore, in practice, we can estimate the sub-ICMs (except \( \mathbf{R}_s \), that is known), calculate their inverse matrices, calculate \( \mathbf{Q}^{-1} \) based on (25), and then obtain the adaptive weighting vector based on (10). This method can save much time and requires fewer training samples.

3.3. Estimation algorithm based on the ICM structure

If the real ICM \( \mathbf{Q} \) satisfies the structure as shown in (24), we can prove that the optimal weighting vector can
be written as a Kronecker product of three weighting vectors, with respect to receiving beamforming, transmitting beamforming, and Doppler filtering. For that purpose, we substitute (25) for $Q^{-1}$ in (8) and then obtain

$$w_{opt} = \frac{(P^{-1} \otimes R_i^T \otimes V^{-1})(c \otimes b \otimes a)}{(c \otimes b \otimes a)^H(P^{-1} \otimes R_i^T \otimes V^{-1})(c \otimes b \otimes a).}$$

This expression is much similar to the MIMO Capon

Kronecker-product structure of the ICM. The same goes for $^Q c_i$.

Step 2: Signals received at multiple snapshots are coherently accumulated with weighting vector $c_i^H P_i^{-1}/(c_i^H P_i^{-1} c_i)$, giving rise to multiple Doppler channels.

In this scheme, even the time-consuming Kronecker-product operation is avoided, and thus the computational cost is further reduced. In fact, based on the Kronecker-product structure of the ICM, this scheme tries to implement the full-scale adaptive estimation algorithm (9) with dimension $MNK$ by a range compressing algorithm and three estimation algorithms with dimensions $M$, $N$, and $K$, respectively.

3.4. Performance evaluation

To evaluate the performance when the ICM $Q$ is adaptively estimated, we define the SINR output for weighting vector $w$ by

$$\text{SINR}_{\text{output}} = \frac{\|\beta\|^2 |w^i d|^2}{w^i E_{\text{zz}} w} = \frac{\|\beta\|^2 |w^i d|^2}{w^i Q w}. \quad (30)$$

For weighting vector (10), the SINR output becomes

$$\text{SINR}_{\text{output}} = \frac{\|\beta\|^2 (d^H Q^{-1} d)^2}{d^H Q^{-1} Q^{-1} d}. \quad (31)$$

The concerned signal processing scheme is based on the Kronecker-product structure of the ICM. Since the sub-ICM $R_i$, can be often considered to be known a priori, only sub-ICMs $V$ and $P$ should be estimated in practice, whose estimates are denoted by $V$ and $P$, respectively. Therefore, the signal processing scheme equivalently estimates the ICM $Q$ by a Kronecker product of the estimates of three sub-ICMs. According to the property of the Kronecker production, the inverse matrix of the estimate of $Q$ can be obtained by

$$Q^{-1} = \hat{P} \otimes I R^T \otimes \hat{V}. \quad (32)$$

The resulting weighting vector can be written as

$$w = \frac{(\hat{P}^{-1} \otimes R_i^T \otimes V^{-1})(c \otimes b \otimes a)}{(c \otimes b \otimes a)^H(\hat{P}^{-1} \otimes R_i^T \otimes V^{-1})(c \otimes b \otimes a)}$$

$$= \frac{(\hat{P}^{-1} c \otimes (R_i b) \otimes (V^{-1} a))}{(c_i^H V a)(b_i^H R_i b)(c_i^H P_i^{-1} c_i)}.$$
The SINR output with respect to the weighting vector (33) becomes

\[
\text{SINR}_{\text{output}} = |\beta|^{-2}(a^H \textsuperscript{V}^{-1} a)^{-1}(b^H R^{-1} c)^2
\]

\[
\quad \times (a^H \textsuperscript{V} \textsuperscript{V}^{-1} a)(c^H \textsuperscript{P}^{-1} c).
\]

(34)

If the ICM \( Q \) is exactly known, the optimum weighting vector is given by (8) and the maximum SINR output can be obtained as [14]

\[
\text{SINR}_{\text{opt}} = |\beta|^2 d^H Q^{-1} d.
\]

(35)

Furthermore, if the real ICM \( Q \) really takes the structure as (24) and the sub-ICMs are exactly known, the maximum SINR output is obtained as

\[
\text{SINR}_{\text{opt}} = |\beta|^2(a^H \textsuperscript{V}^{-1} a)(b^H R^{-1} c)(c^H \textsuperscript{P}^{-1} c).
\]

(36)

To evaluate the performance loss due to inaccurate estimation of \( Q \), the SINR loss factor is a widely used measurement [18], which is defined by

\[
\rho = \frac{\text{SINR}_{\text{output}}}{\text{SINR}_{\text{opt}}} = \frac{(d^H Q^{-1} d)^2}{(d^H Q^{-1} d)(d^H Q^{-1} d)}.
\]

(37)

If the real ICM \( Q \) satisfies the Kronecker-product structure and its estimate is obtained by (32), according to (34) and (36), it can be easily proved that the total SINR loss factor can be expressed by a product of two SINR loss factors, i.e.,

\[
\rho = \rho_a \rho_c,
\]

(38)

where

\[
\rho_a = \frac{(a^H \textsuperscript{V}^{-1} a)^2}{(a^H \textsuperscript{V}^{-1} a)(a^H \textsuperscript{V} \textsuperscript{V}^{-1} a)}
\]

(39)

and

\[
\rho_c = \frac{(c^H \textsuperscript{P}^{-1} c)^2}{(c^H \textsuperscript{P}^{-1} c)(c^H \textsuperscript{P} \textsuperscript{P}^{-1} c)}
\]

(40)

denote SINR losses due to inaccurate estimation of \( V \) and \( P \), respectively. Known a priori, \( R \) does not induce an SINR loss.

It has been proved in [19] that in order to keep the SINR loss lower than 3 dB, as many as \( 2MNK \) independent samples are required to directly estimate the ICM \( Q \). However, if the ICM \( Q \) takes the expected structure and is estimated by a Kronecker product of sub-ICMs, a result of interest is that the training sample number required can be significantly reduced. For instance, from (32), only \( 2N \) training samples are required for \( \rho_a \) to be lower than 3 dB, and \( 2K \) samples for \( \rho_c \) to be lower than 3 dB.

Moreover, adaptive estimation algorithm relying on this ICM structure requires much lower computational complexity. From (10), the inverse matrix of \( Q \) is needed to calculate the weighting vector, and the computational complexity to inverting \( Q \) is as high as \( O(M^2 N^2 K^3) \). However, once the ICM \( Q \) takes the Kronecker-product structure, the inverse matrix of \( Q \) can be obtained in a way as (32). In this case, we just need to invert \( V \) and \( P \), whose computational complexity are \( O(N^3) \) and \( O(K^3) \), respectively. The computational complexity of the Kronecker production is as high as \( O(M^2 N^2 K^3) \), but the signal processing scheme shown in Fig. 1(a) indicates that this operation can be avoided. Therefore, the total computational complexity to obtain inverse matrices of sub-ICMs is just \( O(N^3 + K^3) \), far lower than \( O(M^2 N^2 K^3) \) used to obtain the inverse matrix of \( Q \).

Considering that the second sub-ICM \( R_s^{-1} L \) in (24) is generally known a priori, the signal processing scheme in Fig. 1(a) can be further simplified. In fact, the weighting vectors computed in step 3, which are actually data independent, can be computed offline, and then the range compression in step 2 and the transmitting synthesis in step 3 can be done together by a range compressor with weighting vector \( S^H b_s^H / (b^H R_s^H b) \). This simplification gives rise to another signal processing scheme shown in Fig. 1(b). Note that the transmitting steering vector with respect to a spatial receiving channel should be consistent with the receiving steering vector of the spatial receiving channel, i.e., they should point to the same spatial direction or spatial region.

The key to obtain the advantages of the proposed signal processing scheme is the Kronecker-product structure of the ICM. In this section, interference signals are studied as a whole. But in practice, interference signals present in received signals are of different types and their statistical characteristic may be different. Therefore, a problem of interest is which kind of radar interferences takes an ICM of a Kronecker-product structure.

4. ICM structure of typical interferences and sub-ICM estimation

In this section, we focus on the ICM structure of four typical radar interferences: white channel thermal noise, white active jamming, inactive point scatterers, and clutter. Next, we will discuss how to select training samples from received signals to estimate unknown sub-ICMs.

4.1. White channel thermal noise

Channel thermal noise can often be considered to be white in a wide frequency range [21]. In fact, an infinite band-width white noise signal is a purely theoretical construction. As if the noise power spectral is wider than the sampling frequency of the radar receiver, channel thermal noise can often be considered to be white. It is easy to verify that white channel thermal noise meets all the assumptions in Section 3 and the \( V \) component of channel noise can be written as \( \sigma_w^2 I_K \), where \( \sigma_w^2 \) denotes the noise variance in a range cell before range compression. The decorrelation speed of noise is quick and thus the \( P \) component becomes the identity matrix \( I_K \). Therefore, the ICM of white thermal noise takes the Kronecker-product structure and has the following form:

\[
Q_n = I_K \otimes R_s^{-1} L / \sigma_w^2 I_K.
\]

(41)

4.2. Active white jamming signal

We consider here active jamming sources illuminating temporally white jamming signals. Similarly, as if its power spectral is wider than the sampling frequency of
the radar receiver, active jamming signal can be considered to be temporally white. In this case, active jamming signals satisfy all the assumptions in Section 3 and take an ICM of that structure. For \( N_i \) independent jamming sources with receiving steering vectors \( \mathbf{a}_1(1), \ldots, \mathbf{a}_i(N_i) \) and powers \( \sigma_i^2(1), \ldots, \sigma_i^2(N_i) \), respectively, the \( V \) component of the jamming signals can be written as

\[
\mathbf{V}_j = \sum_{k=1}^{N_i} \sigma_i^2(k) \mathbf{a}_i(k) \mathbf{a}_i^H(k) = \mathbf{A}_i \Sigma_i \mathbf{A}_i^H, \tag{42}
\]

where \( \mathbf{A}_i = [\mathbf{a}_1(1), \ldots, \mathbf{a}_i(N_i)] \), and \( \Sigma_i = \text{diag}\{\sigma_i^2(1), \ldots, \sigma_i^2(N_i)\} \) denotes a diagonal matrix with diagonal elements of \( \sigma_i^2(1), \ldots, \sigma_i^2(N_i) \). The \( P \) component sub-ICM equals the identity matrix \( \mathbf{I}_K \) as well. Combining all the sub-ICMs gives the following jamming ICM expression:

\[
\mathbf{Q}_J = \mathbf{I}_K \otimes \mathbf{R}_S^{-T} / L \otimes \mathbf{V}_j. \tag{43}
\]

### 4.3 Inactive point scatterer

An inactive point scatterer may be a real target biased from the direction of interest, a piece of cloud or earth, or a manmade interference object. The signal signature of an inactive point scatterer is actually similar to a radar target whose signature is given by (6). If \( N_i \) inactive point scatterers are present with statistically independent scattering amplitudes of \( \beta_i(1), \ldots, \beta_i(N_i) \), the covariance matrix of the inactive point scatterers can be written as

\[
\mathbf{Q}_I = \sum_{k=1}^{N_i} |\beta_i(k)|^2 \mathbf{d}_i(k) \mathbf{d}_i^H(k)
\]

\[
= \sum_{k=1}^{N_i} \beta_i^2(k) \mathbf{c}_i(k) \mathbf{c}_i^H(k) \otimes \left[ \mathbf{b}_i(k) \mathbf{b}_i^H(k) \right] \otimes \left[ \mathbf{a}_i(k) \mathbf{a}_i^H(k) \right]
\]

\[
= (\mathbf{C}_i \otimes \mathbf{B}_i \otimes \mathbf{A}_i) \Sigma_i (\mathbf{C}_i \otimes \mathbf{B}_i \otimes \mathbf{A}_i)^H, \tag{44}
\]

where \( \otimes \) denotes the Khatri-Rao product [22], \( \mathbf{A}_i = [\mathbf{a}_i(1), \ldots, \mathbf{a}_i(N_i)], \mathbf{B}_i = [\mathbf{b}_i(1), \ldots, \mathbf{b}_i(N_i)], \mathbf{C}_i = [\mathbf{c}_i(1), \ldots, \mathbf{c}_i(N_i)], \Sigma_i = \text{diag}\{\beta_i^2(1), \ldots, \beta_i^2(N_i)\}, \) \( \mathbf{a}_i(k), \mathbf{b}_i(k), \mathbf{c}_i(k), \) and \( \mathbf{d}_i(k) = \mathbf{c}_i(k) \otimes \mathbf{b}_i(k) \otimes \mathbf{a}_i(k) \) denote the signal signatures of the \( k \)th inactive point scatterer in the corresponding dimensions. For the MIMO beamforming problem, the covariance matrix of inactive point scatterers is proved in [3] to take the following expression:

\[
\mathbf{M}_i = (\mathbf{B}_i \otimes \mathbf{A}_i) \Sigma_i (\mathbf{B}_i \otimes \mathbf{A}_i)^H. \tag{45}
\]

From (44), if just one inactive point scatterer is present, we have

\[
\mathbf{Q}_I = |\beta_i(1)|^2 \mathbf{c}_i(1) \mathbf{c}_i^H(1) \otimes \left[ \mathbf{b}_i(1) \mathbf{b}_i^H(1) \right] \otimes \left[ \mathbf{a}_i(1) \mathbf{a}_i^H(1) \right]. \tag{46}
\]

The ICM takes the Kronecker-product structure, different from the structure (24) in the second sub-ICM however. But if more than one independent scatterer is present from different directions, the ICM does not take the Kronecker-product structure anymore. The same goes for the ICM \( \mathbf{M}_i \).

In fact, the reason why inactive point scatterers do not take the ICM structure as (24) is that it violates the second assumption made for signals before range compression. Before range compression, signals reflected from inactive point scatterers may spread over several range bins.

Reflected from the same object, these signals are actually not statistically independent and then do not satisfy the Kronecker-product structure of the ICM anymore.

### 4.4 Clutter

Clutter can be considered as a group of inactive point scatterers spreading continuously over a spatial region. Therefore, due to the same reason, clutter returns do not take an ICM of the Kronecker-product structure. With a clutter patch represented by \( N_c \) small point scatterers, the clutter covariance matrix can be expressed by [15,23]

\[
\mathbf{Q}_C = \sum_{k=1}^{N_c} |\beta_c(k)|^2 \mathbf{d}_c(k) \mathbf{d}_c^H(k), \tag{47}
\]

where \( \beta_c(k) \) and \( \mathbf{d}_c(k) \) denotes the complex scattering amplitude and the signal signature of the \( k \)th clutter patch, respectively.

### 4.5 Sub-ICM estimation

In a radar receiving channel, channel noise is always present. If other types of interference are present simultaneously, they can be considered to be mutually independent and the whole ICM can be considered to be the sum of their ICMs. For instance, if channel noise is present along with active white jamming signals, the ICM can be written as

\[
\mathbf{Q} = \mathbf{Q}_a + \mathbf{Q}_J = \mathbf{I}_K \otimes \mathbf{R}_S^{-T} / L \otimes \mathbf{V}_j + \mathbf{I}_K \otimes \mathbf{R}_S^{-T} / L \otimes \mathbf{I}_N \sigma_n^2
\]

\[
= \mathbf{I}_K \otimes \mathbf{R}_S^{-T} / L \otimes (\mathbf{V}_j + \sigma_n^2 \mathbf{I}). \tag{48}
\]

From (48), the ICM of radar interferences still holds the Kronecker-product structure. In this case, even the \( P \) component becomes a known identity matrix. However, because inactive scattering sources break this structure, their presence will destroy this structure. Therefore, we just consider active jamming signals and channel thermal noise here, and the impact of inactive points scatterers on algorithms based on this ICM structure will be analyzed later.

Without inactive scattering sources, according to (48), the \( P \) component sub-ICM becomes the identity matrix \( \mathbf{I}_K \) and we just need to estimate the \( V \) component. With an estimate denoted by \( \hat{\mathbf{V}} \), the estimates of \( \mathbf{Q} \) and \( \mathbf{M} \) can be obtained by

\[
\hat{\mathbf{Q}} = \mathbf{I}_K \otimes \mathbf{R}_S^{-T} \otimes \hat{\mathbf{V}}. \tag{49}
\]

and

\[
\hat{\mathbf{M}} = \mathbf{R}_S^{-T} \otimes \hat{\mathbf{V}}, \tag{50}
\]

respectively. In fact, it is the method to estimate the \( \mathbf{V} \) component that makes many MIMO beamforming algorithms different. For instance, the least square beamformer [6] uses \( \mathbf{V} = \mathbf{I} \); the MIMO Capon beamformer [6] uses

\[
\hat{\mathbf{V}}_{\text{Capon}} = \mathbf{X}(k) \mathbf{X}(k)^H / L. \tag{51}
\]

The MIMO APES beamformer [6] uses

\[
\hat{\mathbf{V}}_{\text{APES}} = \hat{\mathbf{V}}_{\text{Capon}} - \frac{\mathbf{X}^H \mathbf{a}^* \mathbf{a}' \mathbf{S} \mathbf{X}^H}{L^2 \mathbf{b}' \mathbf{R}_S \mathbf{b}^*}. \tag{52}
\]
In [6], training samples to estimate $\hat{\mathbf{V}}_{\text{Capon}}$ and $\hat{\mathbf{V}}_{\text{APES}}$ are chosen from received signals before range compression and the sample number is fixed as the subpulse number $L$. In practice, we can also choose samples from range compressed signals and the number of training sample can be chosen more flexibly. Training samples used to estimate an ICM should be statistically independent. But range compressed signals may be statistically correlated if $\mathbf{R}_s$ is not an identity matrix, according to (16). The mutual correlation matrix between range compressed signals is actually known as $\mathbf{R}_s^T \mathbf{R}_s$. To whiten range compressed signals, consider the signal below

$$
\mathbf{z}(k) = \text{vec}(\mathbf{Z}(k) \sqrt{\mathbf{R}_s^{1/2}}) = (\sqrt{\mathbf{R}_s^{T/2}} \otimes \mathbf{I}) \text{vec}(\mathbf{Z}(k)) = (\sqrt{\mathbf{R}_s^{T/2}} \otimes \mathbf{I}) \mathbf{z}(k) \quad (53)
$$

whose covariance matrix is

$$
E[\mathbf{z}(k) \mathbf{z}^H(k)] = (\sqrt{\mathbf{R}_s^{T/2}} \otimes \mathbf{I})(\mathbf{R}_s^{-1} / L \otimes \mathbf{V})(\sqrt{\mathbf{R}_s^{T/2}} \otimes \mathbf{I}) = \mathbf{I} \otimes \mathbf{V}.
$$

(54)

Therefore, the columns of $\mathbf{z}(k) \sqrt{\mathbf{R}_s^{1/2}}$ are statistically independent and have the same ICM $\mathbf{V}$. Now we can obtain an estimate of the $\mathbf{V}$ component by

$$
\hat{\mathbf{V}}_{\text{RC}} = \frac{L}{MN} \sum_{k=1}^{N_s} \mathbf{z}(k) \mathbf{z}^H(k) = \frac{1}{MN} \sum_{k=1}^{N_s} \mathbf{Z}(k) \mathbf{Z}^H(k),
$$

(55)

where $N_s$ denotes the training sample number. From (55), $N_s$ range compressed signals can provide equivalently $MN$ samples to estimate $\hat{\mathbf{V}}_{\text{RC}}$, but $LN_s$ signals before range compression are involved, which can provide $LN_s$ independent samples to estimate $\hat{\mathbf{V}}_{\text{Capon}}$. But it does not mean that range compression decreases the training samples available. Signals in $N_k$ range bins before range compression can provide $N_s$ independent samples for the MIMO Capon beamformer. After range compression, it can contribute about $M(N_k - L + 1)$ independent samples to estimate $\hat{\mathbf{V}}_{\text{RC}}$, if $\mathbf{R}_s$ is full rank. As $N_k \to \infty$, range compression increases the training sample number by about $M$ times.

Because of the aforementioned advantages, we prefer in practice the signal processing scheme based on the algorithm relying on the Kronecker-product structure of the ICM, such as the scheme in Fig. 1(b). However, once inactive scattering sources are present, interference in received signals does not have a Kronecker-product structured ICM anymore. Therefore, a problem of interest now is how an algorithm relying on the ICM structure would perform when the real ICM does not satisfy the expected structure anymore.

5. Influences of inactive point scatterers

For conventional phased-array radar transmitting a thumb-pin shaped beampattern, inactive point scatterers from unwanted directions may have strong powers and hence may impose a greater impact.

However, the presence of inactive scattering sources would destroy the Kronecker-product structure of the ICM and influence the performance of the algorithm relying on that ICM structure. Therefore, in this section, the impacts of inactive point scatterers on algorithms relying on the Kronecker-product of the ICM will be studied through the MIMO beamforming problem. We use the MIMO beamforming problem because just two sub-ICMs are involved and many algorithms are available to make a comparison.

5.1. MIMO beamforming algorithms concerned

In the MIMO beamforming problem, the task is to find a weighting vector $\mathbf{w}$ in accordance to certain criterion. Most of the beamforming algorithms concerned later obtain weighting vectors of the following form:

$$
\mathbf{w} = \frac{\mathbf{M}^{-1}(\mathbf{b} \otimes \mathbf{a})}{(\mathbf{b} \otimes \mathbf{a})^H \mathbf{M}^{-1}(\mathbf{b} \otimes \mathbf{a})}.
$$

(56)

The difference between the beamforming algorithms concerned later is the method to estimate the ICM $\mathbf{M}$. From (44) and (48), if $N_i$ inactive point scatterers are present, the real ICM $\mathbf{M}$ can be written as

$$
\mathbf{M} = \mathbf{R}_s^T / L \otimes (\mathbf{V}_1 + \sigma_n^2 \mathbf{I}_N) + \sum_{k=1}^{N_i} |\mathbf{b}_i(k)|^2 \mathbf{a}_i(k)^H \mathbf{a}_i(k)^H + \text{diag}(\mathbf{b}_i \mathbf{a}_i) \Sigma (\mathbf{b}_i \mathbf{a}_i)^H.
$$

(57)

In [6], signals before range compression are used to estimate the sub-ICM $\mathbf{V}$. However, inactive point scatterers present in different range bins may be different but beamforming is often done at each range bin. Therefore, in order to avoid interference from inactive point scatterers in range bins around the range bin of interest, we always estimate the ICMs by using signals after range compression here.

The first beamformer concerned is termed as the full-scale sample matrix inverse (FSMI) algorithm, which directly estimates the ICM $\mathbf{M}$ by

$$
\hat{\mathbf{M}}_{\text{FSMI}} = \sum_k \mathbf{z}(k) \mathbf{z}^H(k).
$$

(58)

In practice, the training samples may be insufficient to obtain a satisfactory estimate. In this case, the ICM estimate may have a low condition number and the robustness of the FSMI algorithms may be poor. To improve the algorithm robustness, a common method is to use the diagonal loading method [24]. We term the FSMI with diagonal loading as loaded SMI (LFSMI) here, which estimates $\mathbf{M}$ by

$$
\hat{\mathbf{M}}_{\text{LFSMI}} = \mathbf{R}_s^T / L \otimes \hat{\mathbf{V}}_{\text{FSMI}}.
$$

(59)

where $\sigma_d^2$ denotes the diagonal loading level.

The third MIMO beamformer is called structured SMI (SSMI), which estimates $\mathbf{M}$ by

$$
\hat{\mathbf{M}}_{\text{SSMI}} = \mathbf{R}_s^T / L \otimes \hat{\mathbf{V}}_{\text{APES}}.
$$

(60)
where $\mathbf{V}_{SSMI}$ is an estimate of the $\mathbf{V}$ component of the sub-ICMs. The SSMI algorithm estimates $\mathbf{M}$ by a Kronecker product of the estimates of the sub-ICMs, regardless of whether the real ICM takes this structure. Different from the SSMI algorithm, the MIMO Capon algorithm [6] estimates the ICM by using received signals before range compression.

Samples to estimate $\mathbf{V}_{SSMI}$ may contain returns of inactive point scatterers. Echoed signals of $N_t$ inactive point scatterers can be written as

$$\mathbf{Z}_k = \mathbf{A}_d \text{diag}[\beta_t(k)] \mathbf{B}_t^H \mathbf{S},$$

(61)

where $\beta_t(k) = [\beta_{t1}(k), \ldots, \beta_{tN_t}(k)]$. After range compression with weighting matrix $\mathbf{SR}_s^{-1}/L$, received signals can be written as

$$\mathbf{Z}_k = \mathbf{S}_k \mathbf{R}_s^{-1}/L = \mathbf{A}_d \text{diag}[\beta_t(k)] \mathbf{B}_t^H.$$

(62)

According to (55), for the inactive point scatterers, the estimate of the $\mathbf{V}$ component can be written as

$$\hat{\mathbf{V}}_s = \frac{1}{MNC} \sum_{k=1}^{N_t} \mathbf{Z}_k \mathbf{R}_s \mathbf{Z}_k^H(k),$$

(63)

whose expectation is

$$\mathbf{V}_s = \frac{1}{M} \mathbf{A}_d \text{diag}[\beta_t(k)] \mathbf{B}_t^H \mathbf{R}_s \mathbf{B}_t^H \text{diag}[\beta_t(k)] \mathbf{A}_d^H.$$  

(64)

Three types of interferences can be assumed to be statistically independent, and then the expectation of $\mathbf{V}_{SSMI}$ can be written as

$$\mathbf{V}_{SSMI} = \mathbf{V}_s + \sigma_w^2 \mathbf{I}_N + \mathbf{V}_t.$$ 

(65)

The diagonal loading method can also be applied to $\mathbf{V}_{SSMI}$ and the resulting estimation algorithm is termed as loaded SSMI (LSSMI) algorithm, which estimates the $\mathbf{V}$ component by

$$\hat{\mathbf{V}}_{LSSMI} = \hat{\mathbf{V}}_{SSMI} + \sigma_w^2 \mathbf{I}_N.$$ 

(66)

The last beamformer concerned here is called the two side beamforming (TSB) [18], which iteratively searches for two weighting vectors with respect to the transmitting dimension and the receiving dimension, and then uses their Kronecker product as the equivalent weighting vector. See [18] for details. To avoid scaling indeterminacy, two weighting vectors are normalized per iteration. With a minor modification that the two side weighting vectors are normalized per two iterations during the searching procedure, we can obtain a new algorithm that estimates the ICM $\mathbf{M}$ by

$$\hat{\mathbf{M}}_{TSB} = \hat{\mathbf{R}}_{TSB} \otimes \hat{\mathbf{V}}_{TSB},$$

(67)

where $\hat{\mathbf{R}}_{TSB}$ and $\hat{\mathbf{V}}_{TSB}$ are estimates with respect to the $\mathbf{R}_s^{-1}/L$ and $\mathbf{V}$ component, respectively. This algorithm has a performance close to that of the original TSB algorithm and thus is also termed as TSB here. From (67), this algorithm also deems that the real ICM can be expressed by a Kronecker product of sub-ICMs. Unlike the SSMI and the LSMI, the TSB algorithm does not consider $\mathbf{R}_s^{-1}/L$ as known but estimates it instead.

5.2. Definition of improvement factor

The performance of concerned algorithms will be evaluated in terms of the improvement factor (IF) defined by the ratio of the output SINR to the input [18]. For the MIMO beamforming problem, the input SINR can be written as

$$\text{SINR}_\text{input} = \frac{|\beta|^2 (\mathbf{b} \otimes \mathbf{a})^H (\mathbf{b} \otimes \mathbf{a})}{\text{tr}(\mathbf{M})} = \frac{|\beta|^2 MN}{\text{tr}(\mathbf{M})},$$

(68)

where tr$(\cdot)$ denotes the trace. For weighting vector $\mathbf{w}$, the output SINR can be computed by

$$\text{SINR}_\text{output} = \frac{|\beta|^2 |\mathbf{w}^H (\mathbf{b} \otimes \mathbf{a})|^2}{\text{E}(\mathbf{w}^H \mathbf{Z} \mathbf{w})} = \frac{|\beta|^2 |\mathbf{w}^H (\mathbf{b} \otimes \mathbf{a})|^2}{\mathbf{w}^H \mathbf{M} \mathbf{w}}.$$ 

(69)

From (68) and (69), the IF can be calculated by

$$\text{IF} = \frac{\text{SINR}_\text{output}}{\text{SINR}_\text{input}} = \frac{\text{tr}(\mathbf{M}) |\mathbf{w}^H (\mathbf{b} \otimes \mathbf{a})|^2}{MN |\mathbf{w}^H \mathbf{M} \mathbf{w}}.$$ 

(70)

5.3. Simulation results

Now consider a colocated MIMO radar system with 7 receiving antennas and 5 transmitting antennas, all half-wavelength spaced. Assume that the noise level has been normalized such that $\sigma_w^2 = 1$. There are two jamming sources located at the normalized angular frequencies of $-0.3$ and $0.3$, with the same jamming noise ratio (JNR) of 40 dB. The following simulation results are all obtained by averaging 100 simulation runs. Although only 6 iterations are claimed to be sufficient for the TSB to converge [18], we will always run 20 iterations for safety.

5.3.1. Scenario 1

In the first scenario, no inactive scattering source is present in received signals. In this case, the ICM satisfies the Kronecker-product structure. We first study the performance of two diagonal loading based algorithms, i.e., the LFSMI and LSSMI. The variations of their IFs with the loading level are shown in Fig. 2, where the numbers in the legend denote the training sample numbers used, and the symbol ‘N’ in the x-axis means no diagonal loading.

As shown in Fig. 2, both the two diagonal loading methods can improve the IF if the loading factor is properly chosen. A loading factor from $-10$ dB to $10$ dB can often improve the IF and we will use $0$ dB later. With sufficient samples, the improvement of the IF is not obvious for diagonal loading methods anymore, as is well known.

The training sample number requirement is an important performance measurement of an adaptive algorithm. In Fig. 3(a), we show the IFs of concerned algorithms versus the training sample number. It can be found that with sufficient training samples, all concerned algorithms can converge to the maximum IF in the scenario 1, but their convergence speeds are different. Algorithms relying on the ICM structure, such as the SSMI and TSB, converge faster than those not, such as the FSMI and LFSMI. Meanwhile, the SSMI and LSSMI converge faster than the TSB. That is because, the SSMI and LSSMI consider the $\mathbf{R}$ component as known, while the TSB algorithm does not.
use this information and tries to estimate it with training samples. The FSMI and LFSMI directly estimate the ICM $\mathbf{M}$ and thus perform worst. Therefore, in this case, the algorithm relying on the Kronecker-product structure of the ICM has a significant advantage of the training sample number requirement over those not.

With $70(2MN)$ independent training samples, the beampatterns of concerned beamformers are shown in Fig. 3(b). From Fig. 3(b), all the algorithms can efficiently form nulls at the jamming locations. Compared to the optimal beampattern, with the same number of training samples, the beampattern of the SSMI is closer than that of the TSB, and the beampattern of the TSB is closer than that of the FSMI.

The algorithms relying on the ICM structure (16) can converge with much fewer training samples than algorithms not. That is because the real ICM satisfies the Kronecker-product structure in this situation. In what follows, we will consider situations in which inactive point scatterers are present and then the real ICM does not take the structure anymore.

5.3.2. Scenario 2

In scenario 2, along with the jamming signals in scenario 1, inactive point scatterers are present at the normalized angular frequencies of $-0.45$, $-0.15$, $0.15$, and $0.45$, with the same signal noise ratio (SNR) of 20 dB. As shown before, the ICM in this situation does not meet the Kronecker-product structure anymore. In what follows, we concentrate on the performances of those algorithms when the real ICM does not satisfy the Kronecker-product structure anymore. Therefore, we do not consider diagonal loading algorithms in what follows. Meanwhile, all the algorithms are fed with sufficient samples, such that those algorithms can reach their limit performances. In this case, the FSMI becomes optimal; the SSMI estimates $\mathbf{M}$ by (65); the TSB method does not have expectation values of $\mathbf{R}_{\text{TSB}}$ and $\mathbf{V}_{\text{TSB}}$ available so far, which is hence fed by 300 independent training samples in order to reach its limit performance.

Fig. 4(a) shows the IFs versus the training sample number for the beamformers. From Fig. 4(a), although the ICM does not satisfy that structure in this situation, all the algorithms can converge to the maximum IF. The beampatterns of the algorithms are shown in Fig. 4(b). It can be found that all algorithms can efficiently form nulls at directions where jamming signals and inactive point scatterers are present.

The SSMI can reach the optimal performance because it can also suppress all the interferences, by receiving beamforming. For the receiving beamforming, the covariance matrix $\mathbf{V}_{\text{SSMI}}$ determines the positions of beampattern nulls. From (65), the column space of $\mathbf{V}_{\text{SSMI}}$ is determined by the jamming space and the space spanned by columns of $\mathbf{A}_i$, if $\mathbf{R}_s$ is full rank. Therefore, as if the column space of $\mathbf{V}_{\text{SSMI}}$ can accommodate the signal space spanned by all the jamming and inactive point scatterers, they can be suppressed by receiving beamforming of the SSMI. In this case, the number of jamming plus independent inactive scatterers should be less than the number of receiving
antennas. Therefore, it can be predicted that when more interferences are present, the SSMI cannot suppress them anymore, as will be shown subsequently.

5.3.3. Scenario 3
In the last scenario, the ICM does not take the Kronecker-product structure. Moreover, the number of interferences is greater than that of receiving antenna. Along with the two jamming sources in scenario 1, six inactive point scatterers are present at the normalized angular frequencies of \(-0.4, -0.2, -0.1, 0.1, 0.2\) and \(0.4\), all with the same SNR of 20 dB. The convergence curves of the beamformers are shown in Fig. 5(a) and the beampatterns in Fig. 5(b).

From Fig. 5(a), with sufficient training samples, the FSMI algorithm can still converge to the maximum IF, but the TSB and the SSMI cannot reach the maximum IF anymore. From Fig. 5(b), the FSMI can suppress all the interferences, whose number is greater than that of receiving antennas. The benefit is a result of the increased degrees of freedom for colocated MIMO radar. As predicted, the SSMI cannot adaptively suppress all the interferences anymore, because the adaptivity capability of the receiving beamforming cannot deal with more interferences than the number of receiving antennas. The TSB can form nulls at the directions of the interferences, but the nulls are not sufficiently deep, making the TSB difficult to reach the maximum IF.

6. Conclusion and discussion
At the background of colocated MIMO radar, we study target parameter estimation problem when the real ICM may be written as a Kronecker-product structure. Under certain assumptions, we show that the ICM can be expressed by a Kronecker product of three sub-ICMs. The algorithm relying on this structure can be implemented at a low sample number requirement and low computational complexity. We further give the ICM structures of typical interferences and find that white channel noise and white jamming signals take ICMs of the Kronecker-product structure, but inactive scattering sources do not. The influences of inactive point scatterers on algorithms relying on the Kronecker-product structure are studied.
via numerical experiments. It is found that although inactive point scatterers can destroy the Kronecker-product structure of the ICM, the algorithms relying on the structure can still suppress inactive point scatterers, as if the number is less than that of receiving antennas.

We expect that the real ICM can take the Kronecker-product structure, such that the signal processing scheme of colocated MIMO radar would be very simple. But inactive scattering sources, which can destroy the structure, is often inevitable. Although we have shown that the algorithm relying on that structure can still work well in presence of inactive scattering sources, the number of inactive scattering sources that can be adaptively suppressed has decreased greatly. That is because according to the ICM structure, the sub-ICM corresponding to the transmitting dimension has been fixed as the covariance matrix of transmitted waveforms, and thus the transmitting dimension does not have the adaptivity capability. In this case, the advantage of colocated MIMO radar over phased-array radar is degraded. The TSB algorithm tries to exploit the adaptivity potential in the transmitting dimension, but it may also suffer from a performance loss and has not been proved to be able to converge to the optimal performance yet. Therefore, although many target parameter estimation algorithms for colocated MIMO radar have been proposed yet, there are still many problems deserving a further research.

**Acknowledgment**

This work is partially supported by the National Natural Science Foundation of China (60901067 and 61001212), Program for New Century Excellent Talents in University (NCET-09-0630), Program for Changjiang Scholars and Innovative Research Team in University (IRT0954), and the Fundamental Research Funds for the Central Universities. The authors would like to thank all the anonymous reviewers for their insight comments.

**References**


