IMPROVEMENTS TO THE ZOHDY METHOD FOR THE INVERSION OF RESISTIVITY SOUNDING AND PSEUDOSECTION DATA

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Abstract—Two practical improvements to increase the rate of convergence and to overcome the stability problem of Zohdy's method for the inversion of apparent resistivity sounding data have been proposed. Zohdy uses the measured data as the starting model and then assumes that the necessary correction vector for a layer resistivity to improve the current model is equal to the logarithmic difference between the corresponding model response and observed apparent resistivity values. To improve the speed of convergence, the logarithmic change in the layer resistivity is multiplied by a scaling factor calculated from the apparent resistivity differences in the previous two iterations. To improve the stability of the inversion, a weighted average of the apparent resistivity differences is used to determine the correction in the resistivity of each layer. Many tests with computer generated and field data show that the modifications make a significant improvement to the inversion; they reduce the computing time needed by a significant amount with the final model being far less sensitive to noise in the data. The modifications are extended to the inversion of pseudosection data from two-dimensional resistivity surveys.

Key Words: Geophysics, Inversion, Regression analysis, Resistivity.

INTRODUCTION

For the past two decades, several techniques have been developed for the automatic inversion of apparent resistivity data. Some of the most successful techniques are based on nonlinear optimization methods. In the inversion of resistivity sounding data, the steepest descent method (Koefoed, 1979) and several variations of the least-squares method (Inman, Ryu, and Ward, 1973; Constable, Parker, and Constable, 1987) have been used with some success. For the two-dimensional (2-D) inversion of resistivity data from electrical imaging surveys (Griffiths and Barker, 1993), techniques based on the least-squares optimization method have also been used widely (Smith and Vozoff, 1984; deGroot-Hedlin and Constable, 1990; Sasaki, 1992). These gradient-based techniques require the calculation of the Jacobian matrix which consists of partial derivatives with respect to the model parameters at each iteration (Lines and Treitel, 1984). The calculation of the partial derivatives can be time consuming, particularly in the 2-D inversion of apparent resistivity data using microcomputers.

An interesting new iterative technique which avoids the calculation of the partial derivatives was proposed by Zohdy (1989) for the inversion of resistivity sounding data. This technique later was extended to the 2-D inversion of resistivity data by Barker (1992) and to magnetotelluric data by Hobbs (1992). Two disadvantages of this technique are its relatively slow convergence and instability when there is appreciable noise in the data. Practical methods to overcome these problems are discussed in this paper.

THE ZOHDY TECHNIQUE

The two distinctive features of this technique are the construction of the initial guess and the calculation of the correction vector which minimizes the differences between the measured and model data. For the inversion of sounding curves, Zohdy (1989) starts by using a model in which the number of layers is equal to the data points of the sounding curve. In the initial guess, the resistivity of each layer is set to be the same as the corresponding sample value of apparent resistivity (Fig. 1). Barker (1989) recognized that the shift factor used by Zohdy was related to the 2-D inversion of resistivity data by Barker (1992) and to magnetotelluric data by Hobbs (1992). Two disadvantages of this technique are its relatively slow convergence and instability when there is appreciable noise in the data. Practical methods to overcome these problems are discussed in this paper.
directly to the median depth of investigation (Edwards, 1977) of the array used for a homogeneous earth model. In many situations, using a constant shift factor which is equal to the median depth of investigation, gives reasonably fast convergence. For the Wenner array, this shift factor is about 0.5 times the electrode spacing.

After determining the optimum shift factor, the resistivities of the layers then are adjusted by using the differences between the logarithms of the calculated and observed apparent resistivity values. The resistivity of a layer is adjusted using the following equation:

\[ c_i(j) = e_i(j), \]  
where \( j \) and \( i \) represent the \( j \)th layer (and \( j \)th spacing) and the number of iterations respectively.

The logarithmic correction vector is given by:

\[ c_i(j) = \log(p_{i+1}(j)) - \log(p_i(j)), \]
where \( i \) equals the key number of iteration step and \( p_i(j) \) represents the resistivity of the \( j \)th layer during the \( i \)th iteration.

Because:

\[ e_i(j) = \log(p_o(j)) - \log(p_e(j)), \]
where \( p_o(j) \) and \( p_e(j) \) represent respectively the \( j \)th observed apparent resistivity value and the \( j \)th calculated apparent resistivity value for the \( i \)th iteration, then the resistivity of the \( j \)th layer for the \( (i + 1) \)th iteration can be given by:

\[ p_{i+1}(j) = p_i(j) \cdot \exp(c_i(j)), \]

\[ = p_i(j) \cdot [\rho_o(j)/\rho_e(j)]. \]  
Note that the Zohdy method basically assumes that the necessary logarithmic correction in the resistivity of a layer is equal to the logarithmic difference between the observed and computed apparent resistivities at the corresponding datum point.

As a simple example of the use of the Zohdy method, Figure 2 shows the results from the inversion of the Wenner-array sounding curve for a two-layer model. The apparent resistivity values for this test model were calculated using the linear filter method (Koefoed, 1979). The depths to the center of each layer in the initial model were determined by
Improvements to the Zohdy method

Two layer model - 0\% noise

Standard Zohdy inversion method

ITERATION 1 \% RMS Error 12.14

Depth of Layers

|---------------|----------------|----------------|--------------|

Figure 2. Inversion of two-layer model (Wenner array) sounding curve with Zohdy method. A—Initial model; B—model obtained by Zohdy method after 5 iterations.

Multiplying the electrode spacing of the corresponding sounding curve datum point by the equivalent depth factor (0.5 for the Wenner array). For the initial model, the resistivities of the layers were set to be the same as the measured apparent resistivity values (Fig. 2A). Whereas the actual model has a sharp boundary at a depth of 9.1 m (N.B. the model depth scale is given on the top edge of the figure), the resistivity of the initial model decreases more gradually with depth. After 5 iterations, the resistivity distribution of the computed model shows a better agreement with the actual model (Fig. 2B). The change in the root mean squared (r.m.s.) error calculated from the logarithmic differences between the computed and measured apparent resistivity values with iteration number is shown in Figure 3. For the noise-free sounding curve, the r.m.s. error decreased rapidly in the first 5 iterations followed by a slower decline. This slow convergence of the Zohdy method can be a significant problem, particularly for the 2-D inversion of data on microcomputers. The calculations for the examples shown in this paper were performed on an IBM PC compatible microcomputer.

It was noted by Zohdy (1989) that for noisy data the inversion process can become unstable. After a number of iterations, the r.m.s. error can increase if the sounding data are contaminated sufficiently by noise. Then, the model can exhibit anomalous layers with unusually high or low resistivity values. Figure 3 also shows the change in the r.m.s. error with iteration number when the Zohdy method is used for the inversion of noisy data. In this example, Gaussian random noise (Press and others, 1988) with an amplitude of 10\% of the apparent resistivity value is added to each datum point of the sounding curve for the
two-layer model. The r.m.s. error starts to increase after the 4th iteration (Fig. 3).

To overcome this problem, Zohdy (1989) proposed that the inversion process be repeated using the calculated apparent-resistivity sounding curve produced by the model with the lowest r.m.s. error. Thus, two runs of the Zohdy inversion method are needed to interpret one set of data. Although this is not a problem with 1-D sounding data where the sounding curves can be computed rapidly with the linear filter

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**Figure 3.** Error curves for inversion of two-layer sounding curve data with no noise and with 10% random noise using Zohdy method.

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**Figure 4.** Error curves for inversion of two-layer sounding curve data (with no noise) using Zohdy method with and without fast convergence modification.
Figure 5. Inversion of two-layer model (Wenner array) sounding curve with 10% random noise. A—Initial model; B—model obtained with (Standard) Zohdy method after 5 iterations; C—model obtained using Zohdy method with smoothing modification after 5 iterations.
method, this is a disadvantage for 2-D apparent resistivity data which use the slower finite-element or finite-difference method for the forward computations. A possibly better approach is to determine the cause of the instability and to avoid it.

**IMPROVEMENTS AND EXAMPLES**

*Improvements to the rate of convergence*

To improve the rate of convergence, the following modification was made to Equation (1) which gives the logarithmic change in the model layer resistivity:

\[ c_i(j) = f_i(j) \cdot e_i(j). \]  

(3)

The multiplication factor \( f_i(j) \) was set initially to 1.0 for the first two iterations. Then, it was modified by comparing the logarithmic differences \( e_i \) for two successive iterations. The equation used to modify the multiplication factor \( f_i \) is given by:

\[ f_i(j) = f_{i-1}(j) \cdot (1.0 + e_i(j)/e_{i-1}(j)). \]  

(4)

In practice, the value of \( f_i(j) \) is limited to between 1.0 and 3.0 to minimize its effect on the stability of the inversion process. It is used only if the difference between \( e_{i-1}(j) \) and \( e_i(j) \) is larger than 0.1%. Furthermore, if the logarithmic difference \( e_i(j) \) for the \( i \)th iteration is larger than that for the previous iteration the multiplication factor \( f_i(j) \) is set back to 1.0. Figure 4 shows the error curve when the modification (labeled "Fast Convergence") is used for the inversion of the sounding curve for the two-layer model. With this modification, the r.m.s. error decreases at a faster rate after the first two iterations (particularly between the 2nd and 3rd iterations). By using the previous multiplication factor, the number of iterations needed to reduce the r.m.s. error to a given value, for example 1.0%, is reduced by about one-third. This multiplication factor is similar to the successive over-relaxation parameter used to accelerate the convergence rate of iterative techniques for solving linear equations (Golub and van Loan, 1989).

Other methods of modifying the multiplication factor, for example by using the ratio of the change in the model layer resistivity to the change in the logarithmic difference of the corresponding datum point, also were investigated. Equation (4) generally gave the best results.

*Improvements to stabilize the Zohdy method*

It was mentioned earlier that the inversion process can become unstable for noisy data. The models obtained using the Zohdy method in the inversion of the two-layer sounding curve with 10% random noise are shown in Figure 5. It was noted earlier that in this situation the r.m.s. error starts to rise after the 4th iteration (Fig. 3). Figure 5B shows the model obtained at the 5th iteration where some of the model layers show significant deviations from the actual resistivity values. The reason for the instability of the Zohdy inversion process can be determined by considering the apparent resistivity values at the 5th and 6th data points as an example. The apparent resistivity value at the 5th datum point has been decreased by the noise added to it while the value at the 6th datum point has increased compared to the original noise-free values (cf. Fig. 2). The Zohdy inversion

![Figure 6](attachment:image.png)

Figure 6. Error curves for inversion of two-layer sounding curve data with 10% noise using Zohdy method with and without different modifications.
Improvements to the Zohdy method

2-D model used by inversion method

Figure 7. 2-D model used by Zohdy-Barker method.

The process tries to decrease the difference between the calculated and observed apparent-resistivity values at the 5th datum point by decreasing the resistivity of the 5th layer regardless of what happens to the differences at the other datum points. When the resistivity of the 5th layer is decreased, the apparent resistivity datum point is also decreased.

Figure 8. A—Apparent resistivity pseudosection for horst model with 0% noise; B—model obtained after 5 iterations with Zohdy-Barker method; C—model obtained with modified Zohdy-Barker method.
resistivity values of the neighboring points also are decreased. This increases the difference at the 6th datum point. The inversion process tries to compensate for this by increasing the resistivity of the 6th layer in an attempt to increase the apparent-resistivity value at the corresponding datum point. This process continues producing an oscillatory-layer resistivity variation with depth which eventually increases the apparent resistivity r.m.s. error. By the 5th iteration (Fig. 5B) resistivity of the 5th layer has decreased to a value which is too low, whereas the resistivity of the 6th layer has become too high. For the same reason, the resistivity of the 11th layer is too low whereas that for the 12th layer is too high.

One method to avoid this instability is to take into account the apparent resistivity differences at the neighboring points when calculating the change in each layer resistivity. This method is termed the “Smoothed Zohdy Method” in Figures 5C and 6. The equation to correct the model resistivity is modified to the following form:

$$c_i(j) = C_1 e_i(j - 1) + C_2 e_i(j) + C_3 e_i(j + 1).$$  \(\text{(5)}\)

Instead of just using the difference at one datum point to calculate the resistivity change for the jth layer, a weighted average of the difference of the jth datum point and the two neighboring points is used. Normally, the values of the weighting coefficients $C_1$, $C_2$, and $C_3$ used are 0.25, 0.50, and 0.25 respectively. In this manner, if the weighted average value of the difference values at the three points is zero, the layer resistivity is not changed. This insures that the change in the model layer resistivity calculated does not increase the overall difference of the three points.

Figure 6 shows the r.m.s. error curves when 10% random noise is added to the sounding-curve data for the two-layer model. For the Standard Zohdy method, the r.m.s. error starts to rise after the 4th iteration. In comparison, the r.m.s. error for the Smoothed Zohdy method continues to decline at a slower rate after the 4th iteration to an asymptotic value. Figure 5C shows the model obtained with the Smoothed Zohdy method at the 5th iteration. In this situation, the effect of the random noise on the resistivity of the layers is more subdued.

The modifications to improve the rate of convergence and to stabilize the Zohdy method can be combined. Figure 6 also shows the error curve for the inversion of the sounding data with 10% noise when these two modifications are combined. Notice the improvement in convergence in this situation.

In order to confirm that the results obtained did not depend on the particular data sets used, the tests were repeated for different layered models and different noise levels. The results were similar with no significant differences being observed.

**Improvements to the Zohdy–Barker method in 2-D electrical imaging**

In the previous section, we have seen that the model used by the 1-D Zohdy method is a 1-D multilayer earth model. The 2-D model used by the Zohdy–Barker method (Barker, 1992) consists of a number of rectangular blocks. The arrangement of
the 2-D rectangular blocks with respect to the pseudosection data points is shown in Figure 7. The number of rectangular blocks is the same as the number of data points. The horizontal location of the center of each block is placed at the midpoint of the array used to measure the corresponding apparent resistivity datum point. The depth of the center of the block is set at the equivalent depth of the array (0.5 times the electrode spacing for the Wenner array). Note that the left and the right edges for the blocks on the left and right sides are extended horizontally to infinity. The bottom edge of the row of blocks at the bottom row is extended vertically downwards to infinity. The top edge of the topmost row of blocks is extended to the surface. The thickness and width of each interior block is 0.5 and 1.0 times the minimum electrode spacing respectively. Each block is mapped onto a corresponding datum point on the resistivity pseudosection in an arrangement which is similar to that used for the 1-D Zohdy method.

As with the 1-D Zohdy method, the resistivity of each block is set initially to be the same as the apparent resistivity value for the corresponding datum point. In the Zohdy–Barker (Barker, 1992) method, the resistivity of a block is changed at each iteration by the following equation:

$$c_i(l, n) = e_i(l, n)$$

where $l$ is the horizontal number of block or datum point starting from the left-hand side of the model, and $n$ is the vertical level of the block or datum point. $e_i(l, n)$ is the logarithmic difference between calculated and observed apparent resistivity values whereas $c_i(l, n)$ is the logarithmic change in the resistivity of the block $(l, n)$ from the $i$th to the $(i + 1)$th iteration.
The finite-difference method (Dey and Morrison, 1979) is used to calculate the apparent resistivity values for the 2-D model in consideration. Some tests already have been made on the inversion of apparent resistivity pseudosections by Barker (1992) who also provides a detailed description of the electrical imaging method. The models treated by Barker include a two-layer model, a faulted block which extends to the surface, and a horst model. However, Barker (1992) did not address the issues of stability and convergence. It is recognized that the problems of slow convergence and instability again exist when the data are contaminated by noise if one uses the Zohdy-Barker method for the 2-D inversion of apparent resistivity data. The modifications made to the 1-D Zohdy method described earlier were extended therefore to the inversion of 2-D apparent resistivity data. Firstly, the convergence rate can be improved by using the following modification:

\[ c_i(l, n) = f_i(l, n) \cdot e_i(l, n). \] (7)

As in the 1-D situation, the multiplication factor \( f_i(l, n) \) is set initially to 1.0 for the first two iterations and then modified by comparing the difference values \( e_i \) for two successive iterations. A similar equation is used to modify the multiplication factor \( f_i \). It is given by:

\[ f_i(l, n) = f_i(l, n) \cdot (1.0 + e_i(l, n)e_i(l, n)). \] (8)

To overcome the problem of instability resulting from noise in the data, a local weighted average of the logarithmic apparent resistivity differences is employed. The equation used is:

\[ c_i(l, n) = C_o e_i(l, n) + C_s \{e_i(l - 1, n) + e_i(l + 1, n) + e_i(l + 1, n - 1) + e_i(l + 2, n - 1) + e_i(l - 2, n + 1) + e_i(l - 1, n + 1)\}. \] (9)

where \( e_i \) is the logarithmic difference at a datum point, \( C_0 \) is the weight of central datum point, and \( C_s \) represents the weight of surrounding points.

The sum of all the weights is normalized to 1.0. Near the edges of the pseudosection, the weight for the missing points is distributed to the remaining values. The weight for the central datum point \( C_0 \) can have a value between 0.5 (with one-half the total weight) and 0.15 (with almost the same value as the surrounding datum points). A value of between 0.20 and 0.30 is used normally in practice. Note that, if the 2-D filter [Eq. (8)] is applied to a 2-D resistivity pseudosection data because of a 1-D layered earth structure, it will give the same results as using a 1-D filter on a sounding-curve data set. In this situation, the equivalent 1-D filter has weights of \( 2 \cdot C_s \), \( C_o + 2 \cdot C_s \), and \( 2 \cdot C_s \). Thus, a 2-D filter with a central weight \( C_o \) of 0.25 is equivalent to a 1-D filter with weights of 0.25, 0.50, and 0.25 when used for the inversion of pseudosection data from a 1-D layered earth structure.

In practice, both the modifications to improve the convergence rate and stabilize the Zohdy-Barker method are used together. In the following discussion, this approach, will be referred to as the modified Zohdy-Barker method.
Tests with the modified Zohdy–Barker method for 2-D data

The results of some tests with the modifications made to the Zohdy–Barker method are given in this section. A horst model (Barker, 1992) is used as an example. The Wenner-array apparent-resistivity pseudosection for this model is shown in Figure 8A. Figure 9 shows the r.m.s. error curves with the standard and modified Zohdy–Barker methods. The modified Zohdy–Barker method converges at a slightly faster rate. Figures 8B and 8C respectively show the model resistivity sections obtained with the standard and modified Zohdy–Barker methods after 5 iterations. The model resistivity distribution shows a progressive sharpening of the sides and top of the anomaly with each iteration. The resistivity values in the general area of the actual horst structure also become increasingly higher. In fact, at the 10th iteration, the model resistivity values near the bottom of the structure are higher than the true value of 100 Ω-m. The Zohdy method works well for this example probably because the apparent-resistivity pseudosection which forms the initial model has a reasonably similar shape to the true model.

Figure 10A shows the apparent-resistivity pseudosection for the same model with 10% random noise. The pseudosection with 10% noise shows moderate distortion in the shape of the contours. The apparent resistivity r.m.s. error curves from the inversion
of this data set are shown in Figure 11. The r.m.s. error for the standard Zohdy–Barker method starts to rise after the 7th iteration, whereas that of the modified Zohdy–Barker method continues to decrease slowly after the 7th iteration to about 9%. The model obtained with the standard Zohdy–Barker method at the 5th iteration (Fig. 10B) shows severe distortions resulting from the noise in the data. The modified Zohdy–Barker method model (Fig. 10C) much less distortion.

As a final example, the inversion of the data set from an electrical imaging survey across a landfill site in Nottinghamshire, England (Barker, 1992) is studied with the modified Zohdy–Barker method. The Wenner array was used for this survey. The apparent-resistivity pseudosection and the models obtained by the standard and modified Zohdy–Barker methods are shown in Figure 12. Figure 12B shows the model resistivity section obtained after the 5th iteration with the standard Zohdy–Barker method; similar results obtained with the modified Zohdy–Barker method are shown in Figure 12C with the outline of landfill drawn for comparison. The models obtained with both methods are similar and agree well with the outline of the landfill based on existing information.

From the smoothness of the measured apparent-resistivity contours (Fig. 12A), there probably is only a moderate amount of random noise present such that the standard Zohdy–Barker method worked reasonably well. The main difference between the two approaches is a high resistivity zone at a depth of about 10 m near the 110-m mark in the model obtained by the standard Zohdy method. This high resistivity zone is the result of slightly higher values at the 1st and 2nd levels near the 110-m mark in the measured apparent-resistivity pseudosection and seems to be subdued in the other model. Both methods show a low resistivity zone beneath the base of the landfill which is probably the result of leachate contamination into the sandstone bedrock which has a higher resistivity than the landfill material (Barker, 1992). Another possible reason why the Zohdy method works well for this data set is that the subsurface resistivity varies in a gradational manner possibly the result of leachate and groundwater seepage into the sides of the landfill. This gives rise to a relatively smooth apparent-resistivity pseudosection which resembles the actual subsurface resistivity distribution.

CONCLUSIONS

Two simple modifications to the Zohdy technique have been introduced. With these modifications, the Zohdy inversion technique for 1-D and 2-D apparent resistivity data are more stable and converges at a somewhat faster rate. It results in models that are smoother and less sensitive to the noise in the apparent resistivity data points. It generally reduces the number of iterations needed to decrease the r.m.s. error to an acceptable level by about one-third. Similar modifications probably can be made in the use of the Zohdy technique for the inversion of other types of geoelectrical data.

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REFERENCES


