Resonance of a Quasi-Zero Stiffness Vibration System Under Base Excitation with Load Mismatch

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Received 8 September 2016
Accepted 23 February 2017
Published 15 March 2017

The primary resonance and 1/3 subharmonic resonance of a quasi-zero stiffness (QZS) vibration system under base excitation with load mismatch are studied in this research. The incremental harmonic balance (IHB) method is applied to obtain highly accurate solutions involving more dynamic behaviors. The effect of the offset displacement mainly caused by overloading on the primary resonance and displacement transmissibility is investigated. The results indicate that the system exhibits a softening characteristic under certain conditions. Although the isolation performance of the QZS system deteriorates, it still outperforms the equivalent linear system for excitation amplitudes that are not too large. The parametric analysis of the 1/3 subharmonic resonance shows that the response is unbounded, and interesting dynamic behaviors can be observed, such as the jump phenomenon. Moreover, the 1/3 subharmonic resonance can be avoided by applying a larger damping or reducing the excitation amplitude to a lower level.

Keywords: Vibration isolator; quasi-zero stiffness; primary resonance; subharmonic resonance; load mismatch; incremental harmonic balance method.

1. Introduction

Recently, nonlinear vibration isolators with quasi-zero stiffness (QZS) have attracted the attention of many scholars because they possess high static stiffness and low dynamic stiffness. Therefore, the QZS vibration isolator can overcome the inherent contradiction between the natural frequency and the static deflection. A QZS vibration isolator can be implemented by combining the linear positive stiffness element and the negative stiffness corrector. The positive stiffness in the direction of motion can be counteracted by the negative stiffness. As a result, with properly designed structural parameters, the QZS property can be achieved at the static equilibrium position. Thus, this type of vibration isolator is suitable for low-frequency vibration isolation.

Alabuzhev et al.\textsuperscript{1} proposed a variety of prototypes of the QZS vibration isolator and provided the design methodology. Carrella et al.\textsuperscript{2,3} performed a detailed analysis
of a QZS vibration isolator constructed with a vertical spring and two inclined springs and proposed the concept of high-static-low-dynamic stiffness (HSLDS). Robertson et al.\textsuperscript{4} built a QZS vibration isolator using magnets as the negative stiffness corrector and provided the design criteria. Zhou and Liu\textsuperscript{5} then developed an HSLDS vibration isolator using a permanent magnet and two electromagnets as the negative stiffness corrector and demonstrated good isolation performance through an experimental study. Le and Ahn\textsuperscript{6,7} applied a nonlinear vibration isolator to a vehicle seat using two symmetric structures as the negative stiffness corrector and comprehensively investigated the vibration isolation performance. Liu et al.\textsuperscript{8} and Huang et al.\textsuperscript{9} developed a QZS vibration isolator using Euler buckled beams as the negative stiffness corrector and analyzed the isolation performance theoretically and experimentally. Sun et al.\textsuperscript{10} and Sun and Jing\textsuperscript{11} designed a nonlinear vibration isolator with a scissor-like structure and comprehensively studied its isolation performance. Then, Wu et al.\textsuperscript{12} further developed a similar vibration isolator that was inspired by the limb structures of animals/insects in motion vibration control. Zhou et al.\textsuperscript{13} and Cheng et al.\textsuperscript{14} analyzed the dynamic characteristics and the isolation performance of a QZS vibration isolator using two cam-roller-spring mechanisms as the negative stiffness corrector. Meng et al.\textsuperscript{15} built a QZS vibration isolator using a disk spring as the negative stiffness corrector and studied the transmissibility performance in detail. Recently, Sun and Jing\textsuperscript{16} further proposed a QZS vibration isolator for multi-directional vibration isolation by applying the scissor-like structures. In addition, Tang and Brennan\textsuperscript{17} analyzed the isolation performance of an HSLDS vibration isolator under shock excitation. Wang et al.\textsuperscript{18} further investigated the isolation performance of a QZS vibration isolator subjected to random excitation. Additional information about the nonlinear vibration isolator was summarized by Ibrahim.\textsuperscript{19}

In the above-mentioned literature, the scholars mainly focused on a symmetric system in which the minimum stiffness corresponds to the static equilibrium position. However, it is difficult to balance the load in an ideal equilibrium position in practice. The situation of load mismatch, such as overload and underload, is more likely to occur due to various factors but receives less attention from scholars. Huang et al.\textsuperscript{20} investigated the dynamic behaviors and isolation performance of a QZS vibration isolator considering load mismatch using the harmonic balance method (HBM). Abolfathi et al.\textsuperscript{21} used the same method to analyze the force transmissibility of a QZS vibration isolator considering a constant force and load mismatch. Zou et al.\textsuperscript{22} obtained the asymmetric solutions of a wire rope vibration isolator using a modified homotopy analysis method. However, some dynamic behaviors cannot be observed using the first-order HBM, and the accuracy of the solution in the resonant region is insufficient. When using higher orders in the HBM or other analytic methods, such as the multiple scales method, the analytical derivation becomes rather difficult.\textsuperscript{23,24} Thus, in this paper, the incremental harmonic balance (IHB) method is applied to obtain the highly accurate solutions of a QZS system subjected to base excitation. In addition to the primary resonance, the 1/3 subharmonic resonance is also likely to
occur under certain conditions, which could result in a worse isolation performance over the higher frequency range.\textsuperscript{25} To the authors’ knowledge, the influences of load mismatch on the 1/3 subharmonic resonance of the QZS vibration isolator have received less attention. Moreover, it is necessary to investigate the effects of excitation amplitude and damping on the occurrence of 1/3 subharmonic resonance and to provide the avoidance condition.

The remainder of this paper is organized as follows: in Sec. 2, a QZS vibration isolator with a scissor-like structure is developed, and the force and stiffness characteristics are obtained using static analysis. Section 3 includes a detailed derivation of the IHB formulation along with the stability analysis. The effects of load mismatch on the primary resonance and the displacement transmissibility are analyzed in Sec. 4. Then, the 1/3 subharmonic resonance is investigated in detail in Sec. 5. The conclusions of this study are summarized in Sec. 6.

2. Modeling of a QZS Vibration Isolator

The physical model of a QZS vibration isolator using a scissor-like structure as the negative stiffness corrector is shown in Fig. 1. The scissor-like structure is constructed with a horizontal spring, connecting rods, hinge axes and brackets. All the connecting rods have the same length $l$. Each end of the connecting rod is connected to either the hinge axis or the bracket. The scissor-like structure is connected to the loading support and the base plate using brackets. The loading support can only

Fig. 1. Physical model of a QZS vibration isolator. (1) loading support, (2) hinge axis, (3) upper bracket, (4) outer connecting rod, (5) inner connecting rod, (6) vertical spring, (7) vertical guide rod, (8) horizontal spring, (9) lower bracket and (10) base plate.
move in the vertical direction due to the existence of vertical guide rods. When the loading support carries a load $m$, the vibration isolator is initially balanced in the static equilibrium position at which point all the connecting rods are in the same horizontal plane. Thus, the vertical springs are compressed by a deflection $\Delta x = mg/k_v$, where $mg$ and $k_v$ denote the gravity of the load and the total stiffness of vertical springs, respectively. Meanwhile, the horizontal spring, with stiffness of $k_h$, is stretched to a prestretch length of $d$. The stretching force is transmitted to the loading support through the brackets, and its direction is opposite to that of the restoring force generated by the vertical springs. When the magnitude of the negative stiffness is equal to that of the positive stiffness, the QZS property is achieved. Generally, this type of negative stiffness corrector can avoid mechanical instability because the fact that the horizontal spring is mainly stretched as the vibration isolator works.

When the loading support is subjected to a force $f$, it deviates from the static equilibrium position by a displacement $x$, as shown in Fig. 2. Notably, the magnitude of the restoring force of the vibrating system is equal to that of the applied force but in the opposite direction. According to the principle of force balance, the relationship between the applied force and the displacement is given by

$$F(x) = f_v + mg - f_1,$$

(1)

where $f_v = k_v(x - \Delta x)$ denotes the vertical spring force and $f_1 = f_h \tan \alpha$ is the force generated by the negative stiffness corrector. Here, $f_h = k_h(d - 2l(1 - \cos \alpha))$ denotes the horizontal spring force, where $\alpha$ is the angle between the connecting rod and the horizontal plane and $\tan \alpha = x/\sqrt{4l^2 - x^2}$. Therefore, the relationship between the applied force and the displacement can be further expressed as

$$F(x) = k_v x - k_h x \left(\frac{d - 2l}{\sqrt{4l^2 - x^2}} + 1\right).$$

(2)

---

**Fig. 2.** Schematic diagram of the static analysis.
Writing Eq. (2) in the nondimensional form yields

\[ f(u) = u - \beta u \left( \frac{\delta - 2}{\sqrt{4 - u^2}} + 1 \right), \]  

(3)

where \( f = F/(k_v l) \), \( u = x/l \), \( \beta = k_h/k_v \) and \( \delta = d/l \).

The nondimensional stiffness of the system can be obtained by differentiating Eq. (2) with respect to \( u \)

\[ k(u) = 1 - \beta \left( \frac{4(\delta - 2)}{(4 - u^2)^{3/2}} + 1 \right). \]  

(4)

The nondimensional restoring force and stiffness of the system as a function of nondimensional displacement are depicted in Fig. 3. The minimum stiffness corresponding to the static equilibrium position decreases as \( \delta \) increases. Note that when \( \delta \) increases to 1, the minimum stiffness becomes zero, which means that the QZS property is achieved. However, when \( \delta \) increases further, the stiffness in the vicinity of the equilibrium position becomes negative. As a result, the system becomes unstable, which is undesirable in engineering practice. The QZS property can be obtained by setting \( k(u = 0) = 0 \)

\[ \delta_{\text{QZS}} = \frac{2}{\beta}. \]  

(5)

If the vibration isolator does not match with the load, overload or underload could occur. As a result, the load will be balanced in a new static equilibrium position that deviates from the ideal position by an offset displacement, as shown in Fig. 4. The nondimensional stiffness corresponding to the new static equilibrium position is no
longer the minimum. The following sections investigate the effects of the offset displacement which is mainly caused by overload on the resonance response of the QZS system.

Considering relatively small oscillations, Eq. (3) can be approximated by a third-order Taylor series with \( u = 0 \) for simplicity

\[
\tilde{f}_a(u) = \gamma u^3, \tag{6}
\]

where \( \gamma = \beta(2 - \delta)/16 \).

The comparison between the approximate and exact restoring force is shown in Fig. 5. The approximation accuracy is dependent on the nondimensional displacement, and the approximate curve matches well with the exact one in the selected displacement range.
3. IHB Method and Stability Analysis

3.1. IHB method

Assuming that the base plate of the QZS system is subjected to a base excitation
\[ z = Z_c \cos \omega t, \]
where \( Z_c \) and \( \omega \) are the amplitude and frequency of base excitation, respectively, the equation of motion for the load with respect to the relative dis-
placement \( y = x - z \) is governed by
\[
ml\ddot{u} + cl\dot{u} + k_vl(\gamma(u + u_0)^3 + \Delta u + u_0) = mg + mZ_c\omega^2\cos\omega t,
\]
where \( u = y/l \) is the nondimensional displacement, \( c \) is the damping coefficient of a linear viscous damper, \( \Delta u = \Delta y/l \) represents the nondimensional static deflection without load mismatch, and \( u_0 = y_0/l \) denotes the nondimensional offset displacement caused by load mismatch, with \( u_0 > 0 \) for overload and \( u_0 < 0 \) for underload. The system satisfies the following equilibrium equation in the new static equilibrium position:
\[
k_vl(\gamma u_0^3 + \Delta u + u_0) = mg.
\]

Substituting Eq. (8) into Eq. (7) and writing the equation in a nondimensional
form yields
\[
\frac{\Omega^2}{V^2} \ddot{\eta} + 2\zeta \frac{\Omega}{V} \dot{\eta} + a_1 u + a_2 u^2 + a_3 u^3 = \gamma \Omega^2 \cos(V\tau),
\]
where \( \omega_0 = \sqrt{k_v/m} \), \( \zeta = c/(2m\omega_0) \), \( \Omega = \omega/\omega_0 \), \( \tau = \omega t/V \), \( \gamma = Z_c/l \), \( a_1 = 3\gamma u_0^2 \), \( a_2 = 3\gamma u_0 \), \( a_3 = \gamma \), the overdots denote derivatives with respect to \( \tau \) and \( V \) is the subharmonic index with \( V = 1 \) for the primary resonance and \( V = 3 \) for the 1/3 subharmonic resonance.

By introducing the transformation of \( q = u + a_2/(3a_3) \), Eq. (9) can be rewritten as
\[
\frac{\Omega^2}{V^2} \ddot{\eta} + 2\zeta \frac{\Omega}{V} \dot{\eta} + b_1 q + b_3 q^3 = b_2 + \gamma \Omega^2 \cos(V\tau),
\]
where \( b_1 = a_1 - a_2^2/(3a_3) \), \( b_2 = a_1 a_2/(3a_3) - 2a_2^3/(27a_3^2) \) and \( b_3 = a_3 \). Writing Eq. (10) in a general form yields
\[
h(\dot{\eta}, \dot{q}, q, \Omega, \tau) = 0,
\]
where \( h(\dot{\eta}, \dot{q}, q, \Omega, \tau) = \Omega^2/V^2\ddot{\eta} + 2\zeta \Omega/V \dot{\eta} + b_1 q + b_3 q^3 - b_2 - \gamma \Omega^2 \cos(V\tau) \).

The first step called incremental procedure is to obtain the incremental equation. Assuming that \( q_0 \) represents an initially approximate vibrating state corresponding to the frequency \( \Omega_0 \), the neighboring state can be denoted by
\[
q = q_0 + \Delta q, \quad \Omega = \Omega_0 + \Delta \Omega,
\]
where \( \Delta q \) and \( \Delta \Omega \) denote small increments.

Substituting Eqs. (12a) and (12b) into Eq. (11) leads to
\[
h(\dot{\eta}_0 + \Delta \dot{\eta}, \dot{q}_0 + \Delta \dot{q}, q_0 + \Delta q, \Omega_0 + \Delta \Omega, \tau) = 0.
\]
Using a first-order Taylor series about the initial state, Eq. (13) can be written as
\[ h_0 + \frac{\partial h}{\partial q} \bigg|_{q_0} \Delta \dot{q} + \frac{\partial h}{\partial \dot{q}} \bigg|_{q_0} \Delta \ddot{q} + \frac{\partial h}{\partial \Omega} \bigg|_{\Omega_0} \Delta \Omega + \text{higher - order terms} = 0, \] (14)
where \( h_0 = h(q_0, \dot{q_0}, \Omega_0, \tau). \) Ignoring the higher order terms can lead to the following incremental equation:
\[ \frac{\partial h}{\partial q} \bigg|_{q_0} \Delta \dot{q} + \frac{\partial h}{\partial \dot{q}} \bigg|_{q_0} \Delta \ddot{q} + \frac{\partial h}{\partial \Omega} \bigg|_{\Omega_0} \Delta \Omega = -h_0. \] (15)
It is assumed that the periodic solution \( q \) and its increment \( \Delta q \) are in the Fourier series form:
\[ q(\tau) = a_0 + \sum_{n=1}^{N} (a_n \cos n\tau + b_n \sin n\tau), \] (16)
\[ \Delta q(\tau) = \Delta a_0 + \sum_{n=1}^{N} (\Delta a_n \cos n\tau + \Delta b_n \sin n\tau), \] (17)
where \( N \) is the number of harmonic terms considered in the limited Fourier series. In these equations, \( a_n \) and \( b_n \) are the Fourier coefficients of \( q(\tau) \), and \( \Delta a_n \) and \( \Delta b_n \) are the Fourier coefficients of \( \Delta q(\tau) \). The accuracy of the solution is dependent on the number of harmonic terms. The greater the number of harmonic terms is, the higher the accuracy of the solution is.

The next step is the harmonic balance procedure. By applying the Galerkin average procedure, Eq. (15) can be transformed to
\[ \int_{0}^{2\pi} \left( \frac{\partial h}{\partial q} \bigg|_{q_0} \Delta \dot{q} + \frac{\partial h}{\partial \dot{q}} \bigg|_{q_0} \Delta \ddot{q} + \frac{\partial h}{\partial \Omega} \bigg|_{\Omega_0} \Delta \Omega \right) \cdot \delta(\Delta q) d\tau 
\] 
\[ = - \int_{0}^{2\pi} h_0 \cdot \delta(\Delta q) d\tau, \] (18)
where \( \delta(\Delta q) \) denotes the variation of \( \Delta q \). Thus, a group of \( 2N + 1 \) linearized algebraic equations can be obtained by substituting Eqs. (16) and (17) into Eq. (18):
\[ \{ R \} = [C] \{ \Delta a \} + \{ Q \} \Delta \Omega, \] (19)
where \( \{ R \} \) is the residual vector, \( [C] \) is the Jacobian matrix, \{Q\} represents the derivative vector corresponding to \( \Omega \) and \( \{ \Delta a \} \) is the incremental vector with respect to the Fourier coefficients. These values are defined as
\[ \{ a \} = [a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N]^T, \] (20a)
\[ \{ \Delta a \} = [\Delta a_0, \Delta a_1, \ldots, \Delta a_N, \Delta b_1, \Delta b_2, \ldots, \Delta b_N]^T, \] (20b)
\[ \{ R \} = \left\{ \begin{array}{c} R_1 \\ R_2 \end{array} \right\}, \] (20c)
The integral relations for various elements contained in \( \{ R \} \), \([ C]\) and \( \{ Q \}\) are reported in the literature.

According to the literature, the \( \Omega \)-incrementation procedure for obtaining the frequency response curves (FRCs) of the QZS system can be implemented by incrementing \( \Omega \) from point to point. As a result, \( \Delta \Omega = 0 \) throughout the iteration process at every point, yielding the following equations:

\[
[C]^{(i)} \{ \Delta a \}^{(i+1)} = \{ R \}^{(i)},
\]

\[
\{ a \}^{(i+1)} = \{ a \}^{(i)} + \{ \Delta a \}^{(i+1)}.
\]

Given an initial guess of \( \{ a \} \), the matrices \( \{ R \} \), \([ C]\) and \( \{ a \} \) will be updated throughout the iteration process. When the norm of \( \{ \Delta a \} \) is less than a permissible value, the iteration process is completed; thus, the periodic solution is obtained. The iteration stopping criteria is given by

\[
\| \{ \Delta a \} \| < \varepsilon,
\]

where \( \varepsilon \) is the permissible value.

### 3.2. Stability analysis

Once the steady state solution is obtained, its stability can be determined by introducing a perturbed solution \( q^* \):

\[
q^* = q + \eta,
\]

where \( \eta \) denotes a small perturbation. Substituting Eq. (23) into Eq. (10) and expanding the nonlinear term in the Taylor series about the periodic solution leads to the following equation:

\[
\frac{\Omega^2}{V^2} \ddot{\eta} + 2\xi \frac{\Omega}{V} \dot{\eta} + (b_1 + 3b_3q^2)\eta = 0.
\]

Rewriting Eq. (24) in a state variable form yields

\[
\{ \dot{U} \} = [H(\tau)]\{ U \},
\]

where

\[
[H(\tau)] = \begin{bmatrix}
0 & 1 \\
-V^2 \frac{b_1 + 3b_3q^2}{\Omega^2} & -2\xi \frac{\Omega}{V}
\end{bmatrix}.
\]
The stability of the periodic solution is determined by the eigenvalues of the transition matrix. If all the moduli of the eigenvalues are less than unity, then the periodic solution is asymptotically stable; otherwise the solution is unstable. A numerical method is applied to obtain the approximate transition matrix because it is difficult to derive the transition matrix analytically. For that purpose, the interval \([0, 2\pi]\) is divided into \(M\) equal intervals with a width of \(\Delta \tau = 2\pi / M\). Inside each of the intervals, the time-varying matrix \([H(\tau)]\) can be replaced by its average value \([H_j]\), \(j = 1, 2, \ldots, M\). For the \(j\)th interval, \([H_j]\) is given by

\[
[H_j] = \frac{1}{\Delta \tau} \int_{\tau_{j-1}}^{\tau_j} [H(\tau)]d\tau.
\] (27)

Therefore, the transition matrix for the whole period can be expressed as

\[
[\Phi] = \prod_{j=1}^{M} e^{[H_j] \Delta \tau}.
\] (28)

4. Primary Resonance

In this section, the number of harmonic terms is given as \(N = 10\), and the index of subharmonic is given as \(V = 1\). The permissible value for the iteration stopping criteria is set at \(\varepsilon = 10^{-6}\), and the number of intervals for stability analysis is chosen as \(M = 300\). Other basic parameters for the QZS vibration isolator are set as \(\beta = 2\) and \(\delta = 1\) unless otherwise stated.

4.1. Frequency response

The FRCs of the QZS system can be obtained by using the IHB method based on the path following procedure reported in a literature, as shown in Fig. 6. \(A_0\) is the...
constant term and equals $a_0$. The maximum and minimum values of the harmonic terms are $A_x = q_x - A_0$ and $A_m = |q_m - A_0|$, respectively, where $q_x$ is the maximum value of the periodic solution and $q_m$ is the minimum value of the periodic solution. The FRCs of $A_x$ and $A_m$ differ significantly in the resonant region because the system is asymmetric. A valley value exists in $A_0$ corresponding to the resonant frequency. When the solution is triple-valued, one of the solutions is unstable. To verify the validity of the IHB method, a numerical simulation based on the Runge–Kutta method is conducted. The results are also shown in Fig. 6 for comparison. It can be observed that the analytical solutions are in great agreement with the numerical ones, indicating that the IHB method is an effective way to solve the nonlinear problem. The solutions located in the range of (0.2, 0.4) cannot be estimated by numerical simulation. This is mainly due to the occurrence of a subharmonic resonance that can be easily verified by numerical simulation.

Figure 7 shows the amplitude of each of the harmonic terms for different frequencies, where $A_i = \sqrt{a_i^2 + b_i^2}$. When the frequency is located in the resonant region, the first three harmonic terms contribute the most to the solution. If the frequency is far away from the resonant region, then the second and third harmonic terms tend to zero. Therefore, the IHB method can accurately obtain the solution in the resonant region.

The effect of the offset displacement caused by overload on the FRCs for two cases of excitation amplitude is shown in Fig. 8. For the case of $z_e = 0.04$, no obvious peak can be observed in $A_x$ and $A_m$ when there is no load mismatch. Then, for the system with load mismatch, the peak amplitude and the resonant frequency may occur in the FRCs. In addition, the difference between $A_x$ and $A_m$ mainly exists over the resonant region. As $u_0$ increases, the resonant frequency and the peak amplitude of $A_x$ and $A_m$ increase to higher values. This is because the system works beyond the minimum dynamic stiffness point. It should be noted that the larger the offset displacement is, the greater the difference between $A_x$ and $A_m$ over the resonant region is.

![Fig. 7. Amplitude of each harmonic terms for various $\Omega$ when $\zeta = 0.02$, $u_0 = 0.3$ and $z_e = 0.12$.](image)
In addition, increasing $u_0$ can lead to the increase of amplitude of $A_0$. Meanwhile, the valley in $A_0$ becomes more obvious. For the system with $\zeta = 0.1$, the trend of the FRCs is similar to that of the previous case with increasing $u_0$. The main difference is that the FRCs bend to the lower frequency when $u_0$ increases to 0.3, which means that the system exhibits a softening characteristic. It is worth noting that the offset displacement has little effect on the response in higher frequencies.

The influence of the excitation amplitude on the FRCs for two cases of offset displacement is shown in Fig. 9. For the case of $u_0 = 0.1$, when $\zeta_e$ increases, the peak amplitudes of $A_x$ and $A_m$ continue to increase, whereas the resonant frequency first decreases and then increases to the higher value. For the FRC of $A_0$, the valley value and the amplitude over a higher frequency range are reduced gradually as $\zeta_e$ increases. When $\zeta_e$ increases to 0.14, the response becomes unbounded, which has been found in the literature. The valley value of $A_0$ tends to zero for the unbounded response. For the case of $u_0 = 0.25$, the system exhibits a softening characteristic as $\zeta_e$ increases to 0.1. If $\zeta_e$ increases further, such as to $\zeta_e = 0.125$, the FRCs first bend to the left and then to the right, exhibiting a softening-to-hardening.
characteristic. Therefore, the softening characteristic occurs for only certain levels of excitation amplitude and load mismatch. If the excitation amplitude is small, the system mainly exhibits a linear characteristic. It is also worth noting that there is an increasing difference between $A_x$ and $A_m$ over the resonant region with increasing excitation amplitude.

The FRCs for various damping ratios are shown in Fig. 10. It can be seen that increasing $\zeta$ suppresses the softening characteristic caused by load mismatch. The amplitude of $A_0$ in the resonant region continues to increase with increasing $\zeta$. If $\zeta$ is large, no obvious peak or valley can be observed in the FRCs. It can also be observed that the larger the damping ratio is, the smaller the difference between $A_x$ and $A_m$ over the resonant region is.

4.2. Absolute displacement transmissibility

The absolute displacement transmissibility is an important index for evaluating the isolation performance of a vibration isolator; this value is defined as the ratio between the amplitude of absolute displacement of the load and the excitation amplitude.
amplitude. In this study, the peak amplitude of $A_x$ is always greater than that of $A_m$ over the resonant region. Therefore, only $A_x$ is considered for the displacement transmissibility. The absolute displacement of the load for the QZS system is given by

$$q_a = q + z_e \cos \tau.$$  \hfill (29)

Then, the absolute displacement transmissibility can be obtained in decibel form

$$T_x = 20 \log \left( \frac{q_{ax} - A_0}{z_e} \right),$$  \hfill (30)

where $q_{ax}$ denotes the maximum value of $q_a$.

The absolute displacement transmissibility for various offset displacements is plotted in Fig. 11. For comparison, the transmissibility of the equivalent linear system is given. For the case of $z_e = 0.04$, when there is no load mismatch, the QZS system is capable of isolating the vibration in the whole frequency range. As $u_0$ increases, both the resonant frequency and the peak transmissibility continue to
increase. As a result, the isolation performance of the QZS system worsens compared to the ideal system. For the system with $\zeta = 0.1$, the system exhibits a softening characteristic when $u_0$ increases to 0.3. The offset displacement has little effect on the isolation performance over a high frequency range. Moreover, the QZS system still outperforms the equivalent linear system regarding peak transmissibility and isolation frequency band.

The effect of the excitation amplitude on the absolute displacement transmissibility for two cases of offset displacement is shown in Fig. 12. For the case of $u_0 = 0.1$, both the peak transmissibility and the resonant frequency first decrease and then increase to the higher value as $z_e$ increases. As $z_e$ increases to 0.14, the transmissibility is unbounded, and the isolation performance is inferior to that of the equivalent linear system. For the system with $u_0 = 0.25$, the trend of the peak transmissibility and resonant frequency is similar to that of the previous case. The main difference is that the QZS system will behave with softening and softening-to-hardening characteristics as $z_e$ increases. It can be concluded that the isolation performance of the QZS system is superior to that of the equivalent linear system when the excitation amplitude is not too large.

Figure 13 shows the absolute displacement transmissibility for various damping ratios. Note that the influences of the damping ratio on the two systems are similar: increasing the damping ratio leads to the reduction of peak transmissibility but deteriorates the isolation performance in higher frequencies. In addition, the jump phenomenon that may occur in the nonlinear system can be avoided by applying a larger damping.

5. 1/3 Subharmonic Resonance

In this section, the number of harmonic terms is given as $N = 12$, the index of the subharmonic is given as $V = 3$, and the other parameters remain unchanged. The
The FRCs of the 1/3 subharmonic resonance can be obtained using the same method, as shown in Fig. 14, where $A_0$ is the constant term and equals $a_0$ and $A_{1/3} = \sqrt{a_1^2 + b_1^2}$ denotes the amplitude of the 1/3 subharmonic resonance. It can be observed that the response of the 1/3 subharmonic resonance is unbounded and that the solution is double-valued over the whole resonant region. However, only the solutions located on the upper branch are stable, as shown in Fig. 14(b). The amplitude of $A_0$ tends to zero as the frequency increases. Notably, the peak amplitude of $A_0$ corresponds to the valley value of $A_{1/3}$. With the purpose of verifying the existence of the 1/3 subharmonic resonance, the phase diagram and the Poincare map for $\Omega = 1$ are obtained by numerical simulation, as shown in Fig. 15. The phase diagrams obtained by the IHB method and numerical simulation are in excellent agreement, demonstrating the validity of the IHB method. The Poincare map further verifies the existence of the 1/3 subharmonic resonance.

The FRCs of the 1/3 subharmonic resonance for various offset displacements are shown in Fig. 16. For clarity, only the frequency range [0.2, 1] is selected. Interestingly,
the minimum frequency and the valley value of $A_{1/3}$ first decreases and then increase to the higher value as $u_0$ increases. When $u_0$ increases to 0.3, the FRC of $A_{1/3}$ bends first to the left and then to the right. In addition, the response of the 1/3 subharmonic resonance is triple-valued near the frequency where the valley value occurs. It is worth noting that the offset displacement has little influence on the amplitude of $A_{1/3}$ in higher frequencies. In addition, the peak amplitude of $A_0$ continues increasing with increasing $u_0$.

The FRCs of the 1/3 subharmonic resonance for various excitation amplitudes are shown in Fig. 17. It can be observed that the 1/3 subharmonic resonance region is expanded accordingly as $z_\varepsilon$ increases. The minimum frequency and the valley value of $A_{1/3}$ decrease with increasing $z_\varepsilon$. Meanwhile, the peak amplitude of $A_0$ continues increasing. The 1/3 subharmonic resonance is more likely to occur with a larger excitation amplitude, leading to worse isolation performance. Therefore, it is necessary to limit the excitation amplitude to avoid the 1/3 subharmonic resonance.
Additional information on the minimum frequency and the valley value of $A_{1/3}=3$ is shown in Fig. 18. It is evident that when $u_0$ varies over a certain range, such as [0, 0.2] for the system with $z_e = 0.11$, both the minimum frequency and the valley value remain almost unchanged. As $u_0$ increases further, the minimum frequency and the valley value begins to decrease. Interestingly, both the minimum frequency and the valley value undergo a sharp reduction and jump to the lower value as $u_0$ increases to 0.32. If $u_0$ is increased further, the minimum frequency and the valley value start to increase almost linearly. When the excitation amplitude is increased to 0.12, the range of $u_0$ in which the minimum frequency and the valley value are nearly constant decreases. In addition, the jump amplitude is reduced significantly. If the excitation amplitude increases further to 0.13, no obvious jumps can be observed in the minimum frequency or the valley value.

In fact, the jump point corresponds to a critical value for the offset displacement. The FRCs of the 1/3 subharmonic resonance when the jump occurs are shown in

Fig. 17. FRCs of the constant term (a) and the harmonic term (b) for various $z_e$ when $\zeta = 0.02$ and $u_0 = 0.15$.

Fig. 18. Minimum frequency (a) and valley value of $A_{1/3}$ (b) as a function of $u_0$ for various $z_e$ when $\zeta = 0.02$. 
Fig. 19. It can be observed that when the offset displacement exceeds the critical value \( u_0 = 0.218 \), the FRC starts to bend first to the left and then to the right, exhibiting a softening-to-hardening characteristic. Therefore, if the characteristics of the system change, the jumps may occur. Moreover, this jump phenomenon is related to the excitation amplitude. A higher level of excitation amplitude can make it easier to change the hardening characteristic to the softening-to-hardening characteristic. As a result, the curves of minimum frequency and the valley value of \( A_{1/3} \) become smoother, and the jumps may disappear.

The influence of the damping ratio on the 1/3 subharmonic resonance is shown in Fig. 20. The 1/3 subharmonic resonance region lessens accordingly when \( \zeta \) increases. In addition, the minimum frequency and the valley value of \( A_{1/3} \) increase to the higher value, while the peak amplitude of \( A_0 \) continues to decrease. Additional information on the minimum frequency and the valley value is shown in Fig. 21. The minimum frequency and the valley value increases rapidly with increasing \( \zeta \). Note
that the larger the damping ratio is, the greater the rate of increase is. Therefore, the
1/3 subharmonic resonance can be avoided by applying a larger damping. When the
damping ratio is determined, reducing the excitation amplitude to a lower level is
also beneficial for avoiding the 1/3 subharmonic resonance.

6. Conclusions

The primary resonance and the 1/3 subharmonic resonance of a QZS vibration
isolator under base excitation considering load mismatch are investigated in this
paper. The periodic solutions for both resonances are obtained using the IHB
method. The parametric analysis of the primary resonance indicates that increasing
the offset displacement leads to the increase of resonant frequency and peak trans-
missibility. As a result, the isolation performance deteriorates compared to the ideal
system. In addition, the system may exhibit softening and softening-to-hardening
characteristics for certain levels of excitation amplitude and load mismatch, whereas
increasing the damping ratio can suppress those characteristics. For a given offset
displacement, the resonant frequency and the peak transmissibility first decreases
and then increase to the higher value as the excitation amplitude increases. Fur-
thermore, a QZS system with load mismatch can still outperform an equivalent linear
system when the excitation amplitude is not too large.

The effects of the offset displacement, excitation amplitude and damping ratio on
the 1/3 subharmonic resonance are investigated. It is demonstrated that increasing
the offset displacement may lead to a result such that the minimum frequency and
the valley value of the 1/3 subharmonic resonance first decrease and then increase to
the higher value. In addition, a jump phenomenon may occur in the minimum fre-
quency and the valley value for certain levels of load mismatch. The FRCs of the 1/3
subharmonic resonance may bend to the higher frequency with a larger offset dis-
placement. Moreover, both increasing the damping ratio and reducing the excitation

Fig. 21. Minimum frequency (a) and valley value of $A_{1/3}$ (b) as a function of $\zeta$ for various $\zeta_o$ when $\omega_0 = 0.15$. 
amplitude can lessen the 1/3 subharmonic resonance region. Therefore, the 1/3 subharmonic resonance may be avoided by applying a larger damping or reducing the excitation amplitude to a lower level.

Acknowledgments

This work was supported by the Funding of Jiangsu Innovation Program for Graduate Education (KYLX15-0256), the National Natural Science Foundation of China (51675262), the Open Project of State Key Laboratory for Strength and Vibration of Mechanical Structures (SV2015-KF-01) and the Fundamental Research Funds for the Central Universities (XZA15003).

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