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Physical Theory for Capillary Flow Phenomena

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From the assumption that the microscopic behavior of the liquid in an unsaturated porous medium is controlled by the physical laws of surface tension and viscous flow, differential equations governing the macroscopic flow in such a medium are deduced. No special pore-shape assumptions are required, but one topological approximation is needed; i.e., that neither isolated drops nor isolated bubbles occur. Several nonessential simplifying assumptions are used; i.e., that the macroscopic properties of the medium, the character of the liquid, and the pressure of the gas are independent of position, time, and direction. The macroscopic equations are obtained in a fully reduced form, permitting comparison between two media—or between two flow systems—that differ only by scaling factors.

A novel feature of this calculation is its prediction that the liquid-transmission and liquid-capacity properties of an unsaturated medium will exhibit hysteresis in their dependences upon the liquid-gas pressure differential, \( p \). The properties of the medium depend upon the pressure history but are invariant to monotonic time-scale distortions of that history. Such time-invariant functionals have been termed by the authors "hysteresis functions," symbolized by the subscript, \( H \), e.g., \( F_H(p) \). Although methods for measuring and describing the characteristics of specific "hysteresis functions" have not yet been developed, the general validity of this analysis can be studied experimentally by testing predictions that are contained in the reduced variables.

I. INTRODUCTION

The general question of movement of fluids within porous media shares with several other fields of applied physics an interest in methods for attacking problems which, though understood in principle, are extremely complicated in practice. The present paper develops a method of attack upon such a problem.

The simplest type of porous flow problem deals with "saturated" media in which all of the pores are completely filled with one homogeneous liquid. For such cases Henry Darcy discovered in 1856 an empirical proportionality between macroscopic flow rate and driving force.\(^1\) By adding a conservation-of-matter condition to Darcy's law written in differential form, Charles Schlichter in 1899 obtained an equation identical in form with the heat flow equation.\(^2\) This has since formed the basis for the successful development of saturated flow technology.

When insufficient liquid is present to fill all of the pores, so that gas or vapor fills the remaining space, the medium is said to be "unsaturated," or the flow is described as "capillary flow." So long as the liquid "wets" the solid, the pressure within the liquid-filled portion of an unsaturated medium must be less than that in the gas-filled portions. This pressure deficit is directly connected with the surface tension and curvature existing on liquid-gas interfaces within the medium. Obviously the average liquid content and the number and size of channels available for flow are related to this pressure deficit in some complicated fashion. As a result, the analysis of unsaturated flow behavior is much more difficult than the analysis of saturated flow.

To make matters worse, capillary flow behavior is characterized by hysteresis effects which are far from negligible, sometimes amounting to a factor of 30.\(^3\)

In spite of these basic difficulties, progress to date in capillary flow analysis has been quite respectable. During the first quarter of the twentieth century many of the qualitative properties of capillary flow were mapped out, and some success was achieved in describing the unsaturated behavior of regularly packed assemblies of uniform spheres by means of surface-tension calculations. At the beginning of the second quarter of the century key work both in theory and in experimental methods was initiated by L. A. Richards. In 1931 Richards proposed\(^4\) that Schlicher's heat-flow type of equation be further generalized for capillary flow by writing the liquid content and the permeability as unspecified and independent functions of the pressure deficit. This proposal, with miscellaneous changes of variable, has formed the starting point for much of the subsequent work with unsaturated media. Its only noticeable limitation to date seems to be its failure to deal with hysteresis effects.

In an actual porous medium the shapes of the solid particles and of the pores between them are virtually endless in variety and complexity. For saturated flow problems this complexity reduces to a single macroscopic number (permeability) which fully characterizes a homogeneous and isotropic medium. For unsaturated flow, more of the microscopic complexity remains after

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one takes a macroscopic viewpoint; nevertheless the macroscopic description of a medium must still be enormously simpler than its microscopic description. Whereas Richards' proposal empirically encompasses this added complexity within two independent functions the present work starts directly from a physical assumption and from this obtains an equation identical with that of Richards except that the two functions have become somewhat more complex "functionals" which can display hysteresis effects. The physical assumption mentioned is simply that the behavior of the liquid within the pores of the medium will be governed by the classical laws of surface tension and viscous flow.

The analytical method employed is to examine the general properties of the two differential equations and of their solutions—properties which must hold true regardless of pore shapes—and then to transform these general properties into macroscopic form. The first step in this process is to show that the solutions of the surface tension equation can be expected to exhibit a discrete or finite multiplicity. When converted into macroscopic terms this multiplicity dictates the particular type of functional required. Thereafter the various similarity properties or scaling factors of the two differential equations are projected to the macroscopic level to obtain a generalized form of Darcy's law with a complete set of reduced variables.

Throughout the discussion a clear distinction between microscopic and macroscopic points of view is required. In the microscopic viewpoint, interface shapes and flow patterns within individual pores are considered, the differential elements employed being small relative to the pore sizes. In the macroscopic viewpoint the situation is reversed; each macroscopic differential element must contain a sufficient number of pores that its pertinent properties—liquid content, effective flow velocity, permeability, etc.—can be suitably described by statistical averages over many pores. Clearly the range of applicability of such an analysis is thereby limited: pore sizes must not approach too closely to molecular dimensions on the one hand, or to the size of the entire flow system on the other. Just how restrictive these limitations may be is a question best explored by experiment, but it seems likely that this approach may be applicable to many of the flow systems of practical interest with media containing particle sizes in the sand and coarse silt ranges.

Basic Assumptions

Several assumptions beyond the essential one mentioned above have been added for convenience; these could presumably have been avoided by a more general formulation in terms of time-and-position dependent tensors. In addition, a general topological approximation has been required. Available evidence gives some reason to expect that this approximation is good except near saturation on a wetting cycle. The assumptions and approximation follow:

1. Properties of the Porous Framework

The geometric properties of the solid material constituting the porous medium, when described macroscopically, are assumed to be independent of position, orientation, and time; i.e., the medium is assumed to be homogeneous, isotropic, and permanent.

2. Properties of the Liquid

The liquid is assumed to be uniform in its surface tension, contact angle, density, and viscosity throughout every portion of a given flow system. At all points it obeys the classical laws of surface tension and viscous flow.

3. Properties of the Gas

The absolute pressure of the gas is assumed to be constant throughout a given flow system. (This simplification avoids the nuisance of introducing more than one definition of liquid pressure into the analysis, and it is compatible with many of the problems of practical interest.)

4. Phase-Connectivity Approximation

It is assumed that all liquid portions are connected together, i.e., that no isolated drops exist; and also that all gas portions are connected, i.e., that no isolated bubbles exist.

II. CHARACTERISTIC BEHAVIOR OF GAS-LIQUID INTERFACES

The detailed microscopic geometry of the liquid-filled space must be known if one is to calculate flow patterns within the liquid. This "microscopic liquid geometry," as we shall call it, is determined by the shapes and positions of the gas-liquid interfaces within the pores. General properties of the shapes of these interfaces will therefore be expected to have their counterparts in general properties of macroscopic flow.

Multiplicity of Solutions of Surface-Tension Equation

The interface geometry within a given medium depends upon three fixed factors, the pore geometry, the surface tension, and the contact angle; and upon one variable factor, the pressure in the liquid relative to the (constant) gas pressure. This relative pressure will hereafter be called "pressure" or "p" to fit the common usage of physics and hydrodynamics.

The classical differential equation obtained by balancing the force due to pressure against the combination of surface curvature and surface tension is

\[ \frac{2}{\rho_m} = \frac{p}{\sigma}. \]  

In this equation, \( \sigma \) is the surface tension (taken positive) and \( 1/\rho_m \) is a surface-curvature property which is known to mathematicians as the mean curvature (taken
pore geometry will make possible a large number of surfaces. For an actual medium the complexity of the solutions of the surface tension equation at a given interface to adjust its position by sliding along solid surfaces. Additional solution possibilities are made served, additional solution possibilities are made possible. This is done qualitatively in Fig. 1. Although the general behavior of the Poisson equation is preserved, additional solution possibilities are made accessible by the "curling up" process, and by the contact-angle boundary condition which allows the interface to adjust its position by sliding along solid surfaces. For an actual medium the complexity of the pore geometry will make possible a large number of solutions of the surface tension equation at a given pressure, surface tension, and contact angle. Nevertheless the behavior of the "uncurling" Poisson equation gives assurance that these will normally be finite in number, that is, that each stable solution will be discrete and definite.

Hysteresis Due to Multiplicity of Solutions

Over a continuous range of pressure, a surface shape which satisfies Poisson's equation will normally vary in a continuous manner with pressure. The same general behavior may also be expected from any one solution of the surface-tension equation. However it can be demonstrated by an example (Fig. 1) that the multiplicity of solutions will not necessarily be constant over the given pressure range. At a certain value of pressure a given solution may reach a condition of unstable equilibrium; beyond this pressure the given solution will not exist. Physically, if one were to vary the pressure slowly with time until the solution which was initially in existence reached its end point, the interface would have no choice but to "cross over" discontinuously to another solution possible at that pressure. The question of which solution would be chosen need not be considered so long as one is willing to admit that the process would be physically deterministic. If this is admitted, then each solution end point may be associated with a definite cross over to another solution. The simplest possible case, i.e., two solutions which cross over to each other at their end points, is shown in Fig. 1.

In three-dimensional space one could, in principle, construct a model showing simultaneously each of the possible stable solutions for the interface surface at one pressure. If a fourth dimension were available to represent pressure, this model could be extended to describe all stable solutions for all pressures in the prescribed range, and to show at each solution end point the proper cross over. Such a four-dimensional model could be used as a "system-map" to follow out any prescribed (continuous) sequence of pressure changes so that starting with a given initial solution of the equation one could determine unambiguously the shape which would represent the interface at the end of the sequence.

Quantitative deductions made from such a model should agree with the behavior of the corresponding physical system provided that the surface tension equation is obeyed by the physical interface, and provided that the bursts of localized flow following each discontinuous cross over to another solution subside rapidly enough that they have no significant effect on subsequent cross overs. The topological connectivity approximation described earlier is associated with the first of these provisions because the pressure differential across the interface of an isolated drop or bubble might differ from that across the continuous interface. The second provision can always be met if changes of pressure are sufficiently slow. In most cases of practical interest, the pores are so small that the readjustment

* The mean curvature is the algebraic average of the curvatures of the two curves on which the surface cuts two mutually-perpendicular planes containing its normal.

Fig. 1. The shape of the gas-liquid interface is determined not only by the differential pressure but also by the previous history of this pressure.
time following a discontinuous jump of the interface must be measured in tenths of a second, while the time for a significant pressure change is measured in minutes or even in days. For such cases the rates of change are undoubtedly slow enough to satisfy the second requirement.

The important conclusion of this discussion can now be drawn, subject only to the foregoing limitations. As the pressure is changed until the end-point pressure for the existing interface shape is attained, the indicated cross-over will occur just as this pressure is reached, regardless of the time rate at which it was approached. Thus, starting from a given condition and proceeding through any specified (continuous) sequence of pressure changes, the final interface shape will depend on the initial conditions and on the pressure sequence, but not on the time required for going through the various portions of this sequence.

This independence of rate means that monotonic distortions of the time scale will have no effect. The behavior can be described as a time-scale invariant dependence. Since the dependence is on pressure history rather than on pressure alone, the interface shape must be described as a "functional" rather than a "function" of pressure. The authors have coined for such a "time-scale invariant functional" the slightly incorrect—but shorter—term "hysteresis function." A subscript $H$ is used to distinguish such hysteresis functions from true functions, thus: $F_H(p)$. Since the interface shape determines the microscopic liquid geometry, which in turn determines such macroscopic properties as liquid content and permeability, the latter must all reflect the characteristic pressure dependence of the former, i.e., they must be hysteresis functions of the pressure.

Individual jumps of interface position have been observed experimentally by Haines using extremely coarse media; macroscopically such jumps are by definition too small and numerous to be observed individually but their combined effect remains in the form of hysteresis. The preceding discussion has indicated that the general mathematical property of this macroscopic hysteresis should be that of a time-scale invariant functional of pressure.

III. MACROSCOPIC EQUATIONS OF FLOW

Before we can capitalize on the preceding section by writing various macroscopic properties as hysteresis functions of pressure, it is necessary first to define these properties in terms of the basic assumptions of this discussion. In particular, the permeability concept must first be deduced from the scaling properties of the viscous-flow equation.

Darcy’s Law from Viscous-Flow Similitude

The Navier-Stokes equation of hydrodynamics specialized to viscous flow conditions can be written

$$ (f - \nabla \cdot p') = \eta \nabla \cdot v', $$

where $f$ represents body forces (gravity, centrifugal force, etc.) per unit volume of liquid, $\eta$ is the liquid viscosity, and $v'$ is the flow velocity of the liquid. The primes denote a microscopic viewpoint, the equation being applied to the flow patterns within the individual pores and channels of the medium. The boundary condition is taken to be $v' \rightarrow 0$ at all solid surfaces or gas-liquid interfaces.

Let us now focus attention upon a macroscopic differential element within a given capillary flow system. If the macroscopic flow pattern within the pores is described at a given time by $v'_s(r')$, $p'_s(r')$ where $r'$ is the microscopic position vector, this pattern must everywhere satisfy (2) and its boundary condition.

Now let a second flow pattern, $v_s(r')$, $p_s(r')$ be constructed in such a way that $v'_s(r') = h v_s(r')$ and $(f_2 - \nabla p_s(r')) = h(f_1 - \nabla p'_s(r'))$. This second pattern will satisfy (2) since on substitution, $h$ cancels out. It will also satisfy the boundary condition since $hv' \rightarrow 0$ wherever $v' \rightarrow 0$.

From these microscopic patterns we can calculate the corresponding macroscopic quantities $\nabla \cdot p$ and $v$, once an averaging process is defined. By choosing to average both $p$ and $v$ over a macroscopic nearly-plane element of area, the conventions of previous capillary flow work are not disturbed; the normal component of $v$ becomes the volume rate of flow per unit area, while $\nabla \cdot p$ becomes the conventional gradient-defining integral applied to $p'$ over the closed surface of a macroscopic volume element.

Now since the microscopic ratio of $v'_s$ to $v'_1$ is the constant $h$, this constant can be factored out of the integral used in obtaining $v_s$, whereupon the macroscopic ratio of $v_2$ to $v_1$ reduces to $h$. Likewise the ratio of macroscopic driving forces is equal to the corresponding microscopic ratio (after taking account of local perturbations in $p'$), i.e., $(f_2 - \nabla p_s)/(f_1 - \nabla p'_s) = h$. Note that the only restriction which was made initially upon $h$ or $f_2$ was that the driving force in the second case be $h$ times that in the first. It has thus been demonstrated that for any macroscopic driving force parallel to that taken for the first case, other detailed solutions to the viscous flow equation can be found in which the macroscopic flow velocity is parallel to $v_1$ and equal in magnitude to $|v_1|$ times the ratio of the magnitudes of the driving forces.

The velocity has not been shown to be in the same direction as the driving force. However if these two vectors are not parallel, the discrepancy represents a

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$^5$ It is well known that small bubbles rising through liquids obey Stoke’s law, acting as though their skins were rigid; if their surfaces were hydrodynamically "free" to move they should rise 50% faster. This observed rigidity is understandable as a surface energy manifestation.

$^\dagger$ To avoid changing the microscopic liquid geometry, the average macroscopic pressure must be the same in both cases. Further, the difference between this value and $p'$ anywhere within the element must be kept small. However $f$ and the macroscopic $\nabla \cdot p$ may differ both in magnitude and direction from their values in the first case.
directional property of the medium which violates the initial assumption of macroscopic isotropy. For exactly the same reason the macroscopic ratio of velocity to driving force must also be independent of the initial flow direction, \( v_1 \). This ratio is then a simple scalar constant which serves to measure the permeability property of the medium. By convention this constant is symbolized “\( K \)” and called the “capillary conductivity,” the term “permeability” being reserved for \( K/\eta \). This proportionality between velocity and driving force is Darcy’s law in differential form. The macroscopic constant \( K \), clearly depends upon—and in principle could be calculated from—the microscopic liquid geometry of the medium. Therefore it must have the same type of pressure dependence as this geometry, i.e., it must be a hysteresis function of \( p, K_H(p) \). Darcy’s law has therefore been obtained in the form

\[
v = K_H(p) (f - \nabla p).
\]

Conservation of Matter Condition

The most direct and convenient way of defining liquid content for the purposes of this discussion is in terms of “volume fraction” of liquid, i.e., as the volume of liquid per unit of total volume (gas, liquid, and solid). This fraction will be given the symbol \( F \); it is obviously dimensionless and is always considerably less than unity. At saturation it equals the fractional void space of the medium. Since \( F \) is calculable directly from the microscopic liquid geometry, it too must be a hysteresis function of pressure, symbolized as \( F_H(p) \).

If we integrate the normal (outward) component of \( \mathbf{v} \) over a closed surface surrounding a macroscopic region, we obtain the net volume rate at which liquid is flowing out of this region. Since liquids are for our purposes incompressible, the net out-flow must be balanced by the rate of decrease of the volume integral of \( F \) over the region. This integrated conservation of matter condition is mathematically equivalent to the differential condition,

\[
\text{div} \mathbf{v} = -\frac{\partial F}{\partial t} = -\frac{dF_H(p)}{dp} \frac{\partial p}{\partial t}.
\]

When \( v \) is eliminated from the two foregoing equations, (3) and (4), a combined equation analogous to that of Richards is obtained. This combined equation is discussed in Appendix I. Partial solutions of this equation for three gravity-free special cases are derived in Appendix II.

IV. REDUCED VARIABLES

In addition to the velocity similitude of the viscous flow law which has just been used to obtain Darcy’s law, other scaling properties of this law and of the surface tension law are available which can be projected to the macroscopic level. The Darcy-law velocity scaling was concerned with quantities which describe the macroscopic flow state within a system while the remaining scale factors will concern quantities which characterize entire flow systems regardless of their flow states. These quantities—which will be called “system parameters” to distinguish them from the “variables,” \( p, v, r, t \)—are viscosity, surface tension, contact angle, one microscopic and one macroscopic characteristic length, and the body force [which, in general, is a permanent macroscopic force field, \( f(r) \)].

Because of the occurrence of two distinct lengths among these system parameters a straightforward application of dimensional analysis does not suffice to disentangle the scaling factors. It is necessary to proceed a step at a time to extract the information directly from the two microscopic equations. The general plan is to replace the various quantities entering the Darcy and continuity equations, (3) and (4), by corresponding reduced quantities which contain combinations of system parameters with no more than one variable. The combinations are so arranged that any two systems for which the reduced medium-properties and the reduced macroscopic boundary conditions are identical must necessarily exhibit identical reduced flow behavior. Such reduced-variable groupings will be indicated by curly brackets.

“Similar” Microscopic Geometries; Microscopic Characteristic Length, \( \lambda \)

Because of the linearity of the two microscopic equations, it is possible to obtain detailed similitude of interface shapes and of microscopic flow patterns between two media whose solid geometries differ only by a constant magnifying factor. Two such media will be called “similar” media, by analogy to the familiar term “similar” triangles. (Henceforth the word “similar” will be italicized to give it this special meaning.) When the interface geometries are also similar, the two media will be said to be “in similar states.” A pair of similar media in similar states is illustrated in Fig. 2. When each of these geometries is “reduced” i.e., expressed in terms of a characteristic length, \( \lambda \), (see Fig. 2) the resulting reduced geometries are identical.

In practice, the occurrence of detailed similarity throughout the microscopic geometries of two media

![FIG. 2. Illustration of two “similar media” in “similar states.”](image-url)
has zero probability. We might argue that since the property will only be employed to obtain macroscopic results, a statistically equivalent criterion should serve equally well. However, this substitution appears merely to transfer our position to the other horn of a dilemma—the problem of stating such a statistical criterion in geometric terms is hopelessly complicated. Fortunately a sound argument can be patched together by combining the two viewpoints. We can derive macroscopic results from the assumption of exact microscopic similarity and then note that these results are necessarily the same as those we would have obtained from our hypothetical statistical criterion. Experimentally it is possible to synthesize media with different \( \lambda \)'s which must closely satisfy the statistical criterion for similarity.

**Similitude of Surface-Tension Equation; Reduced Pressure, \( \{ \lambda p/\sigma \} \)**

A necessary condition for two similar media to be in similar states is that the reduced film curvatures must be the same. Since curvature is a reciprocal length, it is reduced by multiplication by \( \lambda \). When this reduction is applied to the surface tension equation, (1), it becomes

\[
2(\lambda/r_m) = \{ \lambda p/\sigma \}.
\]

From this it is clear that two media in similar states must have equal values of \( \{ \lambda p/\sigma \} \) and that \( \{ \lambda p/\sigma \} \) is therefore the desired reduced-pressure grouping.

A second condition for similarity of state is that the contact angle at each point in one medium must match the contact angle at the corresponding point of the other medium. This limits comparisons between systems to those having the same contact-angle characteristics. In subsequent comparisons this contact angle correspondence will always be assumed to hold.

If we now repeat in terms of reduced geometry and reduced pressure the earlier discussion regarding multiplicity of solutions, using Eq. (5) instead of (1), it is obvious that the reduced solutions for two similar media with the same contact angle will be identical in all respects including the cross overs at solution end points. It is also apparent that if two such media are started from the same reduced state and carried through the same sequences of reduced pressure, the reduced end states will be identical. Accordingly the reduced microscopic liquid geometries of two similar media must be identical hysteresis functions of reduced pressure.

**Reduced Functionals; ** \( \{ F_H(\lambda p/\sigma) \} \) and \( \{(\eta/\lambda^3)K_H(\lambda p/\sigma)\} \)

The foregoing statement leads directly to reduced expressions for the two functionals which characterize media.

The liquid volume fraction, \( F \), being a ratio of volumes, can be calculated just as well from reduced liquid geometry as from actual geometry. Two media having the same reduced liquid geometry must therefore have identical values of \( F \). Combining this with the foregoing reduced-pressure dependence of reduced microscopic liquid geometry it is seen that two similar media must be represented by identical functionals, \( \{ F_H(\lambda p/\sigma) \} \).

The conductivity, \( K \), can also be calculated from the microscopic liquid geometry, but whereas \( F \) can be calculated solely from the reduced geometry, \( K \) is also sensitive to the geometric scale factor, \( \lambda \), and to the viscosity, \( \eta \). Since \( K \) is a ratio of driving force to velocity, it is apparent from the viscous flow equation, (2), that \( K \) must vary inversely as \( \eta \); and since the operator \( \nabla^2 \) is a reciprocal length squared, \( K \) must also vary directly as \( \lambda \). Thus two similar media in similar states must have identical values of \( \{ \eta K/\lambda^3 \} \).

Adding now the reduced pressure-dependence argument used for \( F \), it is seen that two similar media must be represented by identical functionals,

\[
\{(\eta/\lambda^3)K_H(\lambda p/\sigma)\}.
\]

**Macroscopic Similitude: Reduced Space, Time, Velocity**

The scaling properties inherent in the two microscopic equations (1) and (2) have now been exhausted, but the remaining reduced variable groupings can never be put together without ambiguity by employing the scaling information already stored in the macroscopic Darcy equation, (3). As the first step, the \( p \) which appears as \( \nabla p \) in this equation must be reduced to \( \{ \lambda p/\sigma \} \) and the \( K \) which appears must be reduced to \( \{ \eta K/\lambda^3 \} \) by multiplying through with appropriate constants, thus

\[
\frac{\eta}{\lambda} \nabla = \frac{\eta}{\lambda^3} K_H(\frac{\lambda p}{\sigma}). \left[ \frac{\lambda}{\sigma} \nabla - \frac{\lambda p}{\sigma} \right].
\]

For the special case, \( f=0 \) this equation shows that only the \( \nabla \) operator (a reciprocal macroscopic length), and \( v \) (macroscopic velocity) are not yet in reduced forms. If one of these is expressed in terms of an arbitrary characteristic value, the other will automatically be determined. A characteristic macroscopic length, \( L \), will therefore be assigned, whereupon the reduced \( \nabla \) will become \( \{ L \nabla \} \) and the final reduced form of Eq. (3) is seen to be

\[
\left[ \frac{nL}{\lambda^2} \nabla \right] \left[ \frac{\eta}{\lambda^3} K_H(\frac{\lambda p}{\sigma}) \right] \left[ \frac{\lambda L}{\sigma} \nabla - \{ L \nabla \} \frac{\lambda p}{\sigma} \right].
\]

The continuity equation becomes,

\[
\left[ \frac{nL}{\lambda^2} \nabla \right] \left[ \frac{\eta}{\lambda^3} K_H(\frac{\lambda p}{\sigma}) \right] \left[ \frac{\lambda L}{\sigma} \nabla - \{ L \nabla \} \frac{\lambda p}{\sigma} \right] = \frac{\partial \{ F_H(\lambda p/\sigma) \}}{\partial \{ (\lambda p/\sigma) \}}.
\]

\( \$ \) Note that the symbol \( \{ F_H(\lambda p/\sigma) \} \) as used here for the reduced functional does not strictly correspond to the original usage for \( F_H(\lambda p/\sigma) \). Regarded as functionals of \( \lambda p/\sigma \) alone, \( \lambda \) and \( \sigma \) being fixed, \( \{ F_H(\lambda p/\sigma) \} = F_H(\lambda) \). A similar statement applies to \( K_H(\lambda) \).
The reduced macroscopic equations, (6) and (7) are seen to be identical in form with their parent equations (3) and (4), each variable in the parent equation having been replaced by a corresponding reduced variable (in curly brackets).

If two flow systems are composed of similar media, and if they are subjected to macroscopic space-time boundary conditions which are identical when expressed in terms of the foregoing reduced variables, it is clear from the reduced differential equations that the entire subsequent reduced behavior of the two flow systems must also be identical. The two systems will therefore be designated as “similar flow systems.” The behavior of two such systems is shown for illustration in Fig. 3 (not genuine data). The only difference between the two sets of curves in this diagram is in the labeling of curves and axes; if these had been labeled in terms of reduced coordinates there would have been no difference whatever. In practice, the technique just described for making comparisons between entire flow systems is perhaps the most immediately useful result of this paper. For reference, a complete list of reduced variables is tabulated in Appendix III.

V. CONCLUDING REMARKS

From the surface tension and viscous flow laws, some simplifying assumptions, and a topological approximation, it has been concluded that the flow properties, $F$ and $K$, of an unsaturated medium will be “hysteresis functions” of pressure. Furthermore the similitude properties inherent in these assumptions have yielded a Darcy’s law expression and a complete set of reduced variables. These results are independent of the microscopic geometry of the porous medium.

The assumptions themselves imply various limitations to the applicability of the results. In a separate pair of papers the authors mention some of these limitations, discuss the potential applicability of the results in the special field of soil physics, and appraise some of the pertinent experimental information already available in the literature. In general, the theory would be expected to work best with coarse materials in the range of sands, and perhaps silts, at moderate liquid contents. Certain of the results can be expected to work over a wider range of conditions than other portions, e.g. the continuity equation will always hold so long as compressibility is negligible, while the Darcy law will probably still work under conditions for which some of the reduced-variable predictions should show serious deviations. Briefly, the experimental information available so far appears generally favorable, sometimes under such conditions that the assumptions would appear to be far from realistic. An adequate estimate of the range of applicability must await the outcome of rather extensive further experimentation.

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APPENDIX I. DETAILS OF COMBINED DIFFERENTIAL EQUATION OF FLOW

The elimination of $v$ from Eqs. (3) and (4) gives

$$\text{div}[K_H(p)(\nabla p - f)] = \frac{\partial F_H(p)}{\partial t}. \quad (8)$$

This can be expanded into the form

$$K_H(p) \text{div}(\nabla p - f) + \frac{dK_H(p)}{dp}(\nabla p \cdot [\nabla p - f]) = \frac{dF_H(p)}{dp} \frac{d\rho}{dt}. \quad (9)$$

In this form three different functionals appear which depend on the medium; $K$, $dK/\rho$, and $dF/\rho$; only two of which are independent. A convenient form for analytical purposes can be obtained by dividing through by $K$ and coining new symbols, $C$ and $D$, for the combined functionals which then appear, thus:

$$\nabla^2 \rho - \text{div} + C_H(p)(\nabla p \cdot [\nabla p - f]) = \frac{(\partial \rho/\partial t)}{D_H(p)} \quad (10)$$

where by definition

$$C_H(p) = \frac{dK_H(p)}{dp} = \frac{d \ln K_H(p)}{K_H(p)} \quad (11)$$

$$D_H(p) = \frac{K_H(p)}{dF_H(p)/dp}. \quad (12)$$

The $D_H(p)$ functional is placed in the denominator to correspond with the standard coefficient of diffusivity, $D$. The second functional $C_H(p)$ does not have an
analog in nonlinear diffusion theory. For steady-flow
cases, the right side of the equation vanishes; showing
that \( C_H(p) \) alone fully describes steady-flow behavior.

The first expanded form, Eq. (9), is identical with
Richards’ equation (6) except that the two functions
used by Richards have become functionals, and the
time invariance of \( K \) has allowed the \( \nabla K \) of Richards’
expression to be expanded into the more tractable form
\( (dK/dp) \nabla p \).

The second expanded form has two further advan­
tages: (1) the range of variation of the functionals with
pressure is markedly reduced in this form—it may even
be possible to approximate \( C_H(p) \) by a constant for
some purposes. (2) from flow system data which is
accurate enough for determination of the required
derivatives, the functionals \( C \) and \( D \) can be evaluated
more directly than can \( K \) and \( F \) (although the latter
can be determined from the former provided the
integration constants are known independently).

**APPENDIX II. SOLUTIONS OF ONE-DIMENSIONAL
FORCE-FREE PROBLEMS**

Consider a semi-infinite one-dimensional problem in
which a uniform medium containing liquid at pressure
\( p_0 \) initially occupies all points of positive \( x \) in a Carte­
sian reference frame. At \( t \geq 0 \) liquid is applied at con­
stant pressure \( p_1 \) to the plane \( x = 0 \). The pressure change
will be monotonic in this case; the hysteresis functions
will behave like simple functions so that subscripts can
be omitted. The differential equation (10) becomes

\[
\frac{dx}{dt} = C(p) \left( \frac{p_1}{D(p)} \right)
\]

where the subscripts denote partial differentiations.
Applying the general Maxwell relation \( l_m m_1 = -1 \),
and also the general relation \( l_m m_2 = -m_1 \) \((m_1)^3 \) allows us
to make \( x \) the dependent variable, giving

\[
\frac{D(p)}{x^2} \left[ \frac{x^{pp}}{x^p} - C(p) \right] = x_t.
\]

This is a variables-separable form in which \( x = T(t) \)
\( \cdot P(p) \). Substituting gives

\[
x = \frac{1}{P(p)}
\]

where \( P(p) \) obeys

\[
P(p) = \left( \frac{2D(p)}{P'(p)} \right) \left( \frac{P''(p)}{P'(p)} - C(p) \right)
\]

and is subject to the boundary conditions

\[
P(p_1) = 0; \quad 1/P'(p_0) = 0.
\]

(The first of these boundary conditions is obvious, the
second comes from the macroscopic spatial continuity
of \( P \) within the medium.)

One can achieve similar results for cylindrical and
spherical cases of one-dimensional flow, except that in
the first case a term \( -P'(p)/P(p) \) and in the second
case twice this amount, is added within the brackets of
the ordinary differential equation for \( P(p) \). Unfor­
tunately, both of the latter solutions have a pole at the
origin and therefore require that \( p_1 \) be infinite for a
nontrivial result. The latter solutions can, if desired, be
tested experimentally in a somewhat academic way by
applying liquid to a small but finite hole size at any rate
proportional to \( \sqrt{t} \) and observing space-time distribu­
tions of pressure at distances from the origin large
compared to the hole size.

It will be noted that all of the foregoing solutions
treat \( p_0 \) and \( p_1 \) as arbitrary; there is no requirement that
\( p_1 \) be zero or that \( p_1 \) be greater than \( p_0 \). In most experi­
ments to date, \( p_1 \) has been zero, and \( p_0 \) has been essen­
tially \(-\infty\).

**APPENDIX III. TABULATION OF REDUCED
VARIABLES**

A dot subscript is used to denote each reduced variable, e.g. \( \dot{p} = (\lambda \rho / \sigma) \). When Eqs. (3) and (4) are
written with all quantities dotted they become Eqs.
(6) and (7), respectively. The combined Eqs. (8), (9),
and (10) can also be written with all quantities dotted.
The following section also serves to define all symbols
used in this paper and to give their cgs units.

**Reduced Microscopic Interface Geometry, \( \lambda (\text{cm}) = \) Characteristic Pore Size**

\[
\begin{align*}
1/\tau_m &= \{\lambda / r_m\} \quad &1/\tau_m &= \text{mean film curvature, (cm}^{-1}).
\lambda &= \{\lambda \rho / \sigma\} \quad &\rho &= \text{pressure excess of liquid relative to gas, (g cm}^{-1} \text{ sec}^{-2}).
\alpha &= \{\alpha\} \quad &\sigma &= \text{surface tension, (g sec}^{-2}).
\end{align*}
\]

**Reduced Flow Properties of Medium**

\[
\begin{align*}
F_{H.}(\dot{p}) &= \{F_{H}(\lambda \rho / \sigma)\} \quad &F &= \text{liquid volume fraction (dimensionless).}
K_{H.}(\dot{p}) &= \left\{ \frac{\eta}{\lambda^2} K_{H}(\lambda \rho / \sigma) \right\} \quad &K &= \text{capillary conductivity} \quad (g^{-1} \text{ cm}^2 \text{ sec}).
\end{align*}
\]

\[
\eta = \text{viscosity (g cm}^{-1} \text{ sec}^{-2}).
\]
\[ C_H(p) = \frac{\sigma}{\lambda} C_H\{\lambda p / \sigma\} \]

\[ C_H(p) = \frac{d \ln K_H(p)}{dp} \quad (g^{-1} \text{cm sec}^2). \]

\[ D_H(p) = \frac{\eta}{\lambda} D_H\{\lambda p / \sigma\} \]

\[ D_H(p) = \frac{K_H(p)}{dF_H(p)/dp} = \text{"capillary diffusivity"} \quad (\text{cm}^2 \text{sec}^{-1}). \]

**Reduced Macroscopic Variables;** \( L(\text{cm}) = \text{Macroscopic Characteristic Length} \)

- \( r. = \{r/L\} \)
  - \( r = \) position vector (cm).
- \( \nabla. = \{L \nabla\} \)
  - \( \nabla = \) gradient operator (cm\(^{-1}\)).
- \( \text{div.} = \{L \text{div}\} \)
  - \( \text{div} = \) divergence operator (cm\(^{-1}\)).
- \( \nabla^2 = \{L \nabla^2\} \)
  - \( \nabla^2 = \) divergence-of-gradient or Laplace operator (cm\(^{-2}\)).
- \( f. = \{L \eta \}\) \( f = \) body force per unit volume (g cm\(^{-2}\) sec\(^{-2}\)).

  For gravity, \( f = \rho g \) where \( \rho = \) density (g cm\(^{-3}\)), and \( g = \) acceleration of gravity vector (cm sec\(^{-2}\)).

  For "centrifugal force" \( |f| = \rho \omega^2 r \) where \( \omega \) is radian angular velocity (sec\(^{-1}\)), and \( r \) is radius (cm).

- \( \psi. = \{L \psi\} \)
  - \( \psi = \) macroscopic flow velocity (cm\(^3\) sec\(^{-1}\)/cm\(^2\)) or (cm sec\(^{-1}\)).

- \( t. = \{L^2 \}\) \( t = \) time (sec).

In the foregoing discussions primes are used on \( v, p, f \) where a microscopic viewpoint is to be specified.