Sharp conditions for the compactness of the Sobolev embedding on Musielak–Orlicz spaces

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Abstract
We prove the compactness of the Sobolev embedding for Musielak–Orlicz spaces by way of simple conditions on the Matuszewska index of the underlying space. We provide counterexamples to show the sharpness of our conditions.

KEYWORDS
Matuszewska index, modular spaces, Musielak–Orlicz spaces, Sobolev embedding, variable exponent p-Laplacian, variable exponent spaces

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1 INTRODUCTION

The importance of embedding-type theorems in different branches of analysis cannot be overemphasized: In particular, Sobolev embedding theorems were recognized early in their history as an invaluable device, especially in harmonic analysis and partial differential equations. On the other hand there is an increasing interest in the study of Orlicz and Musielak–Orlicz spaces in parallel to that of Lebesgue spaces of variable exponent in the spirit of [5,8]. From the early stages of the theory of variable Lebesgue spaces it was realized that, under certain conditions on the variable exponent \( p(\cdot) \) and for a bounded domain \( \Omega \) the embedding

\[
W^{1,p(\cdot)}_0(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)
\]

is compact (see [8]). In particular, Poincaré-like inequalities still hold for variable exponent spaces and such spaces are suitable for the study of solvability (existence, uniqueness and regularity of the solutions) of partial differential equations, notably for systems of differential equations arising in the analysis of electrorheological fluids and in certain problems related to image processing (see [2–5,8]). In particular, the validity of Poincaré’s inequality opens the door to the study of the variational first eigenvalue of the \( p(\cdot) \)-Laplacian (which arises naturally as the derivative of the corresponding modular, see [6,9,10]).

The class of variable-exponent Lebesgue spaces is but a particular instance of a much more general class of functions, whose most remote reference in the literature is to be traced back to Nakano, [13]: The Musielak–Orlicz class, for the precise definition of which we refer the reader to the body of the article. In this work we aim at studying conditions on the Musielak–Orlicz function \( \varphi \) under which the embedding

\[
W^{1,\varphi}_0(\Omega) \hookrightarrow L^{\varphi}(\Omega)
\]

is compact. Here \( L^{\varphi}(\Omega) \) and \( W^{1,\varphi}_0(\Omega) \) stand for the Musielak–Orlicz class and the Sobolev–Musielak–Orlicz class (i.e., the class of functions in \( L^{\varphi}(\Omega) \) whose weak derivatives belong to \( L^{\varphi}(\Omega) \)), respectively. Our results generalize those of Hudzik [7].

Associated to every Musielak–Orlicz function (MO), $\varphi$, the so-called Matuszewska–Orlicz index of $\varphi$ (see [11]) generalizes the role of the exponent $p$ in the classical Lebesgue spaces; in particular the exponent $p$ is easily verified to be the Matuszewska–Orlicz index of the MO function given by

$$(x, t) \mapsto t^{p(x)}.$$ 

In this spirit we observe that sharp conditions on the Matuszewska–Orlicz index of the MO function $\varphi$ guarantee the compactness of the embedding (1.1). Our condition is trivially satisfied by the exponent $p$ for the Sobolev embedding stated in [8] and proved in detail in [6]. We provide examples to the effect of the sharpness of our condition. Furthermore, we show that there is a gain in the integrability in the sense that there exist a variable exponent $m(x)$ defined on $\overline{\Omega}$ and positive constants $C$ and $\theta$ such that for large enough $t$ one has

$$\varphi(t, x) \leq Ct^{m(x)+\theta}$$

and that there is a compact embedding

$$W_0^{1,\varphi}(\Omega) \hookrightarrow L^{m(x)+\theta}(\Omega) \subset L^\varphi(\Omega).$$

We refer the reader to the body of the paper and Corollary 5.2 for the specifics.

2 | MUSIELAK–ORLICZ SPACES

Throughout this paper $\Omega \subset \mathbb{R}^n$, $n \geq 1$, will stand for a bounded, Lipschitz domain. A convex, left-continuous function

$$\varphi : [0, \infty) \to [0, \infty)$$

with $\varphi(0) = 0$, $\lim_{x \to \infty} \varphi(x) = \infty$ and $\lim_{x \to 0^+} \varphi(x) = 0$ will be said to be an Orlicz function. In particular, any Orlicz function is non-decreasing. The term generalized Orlicz function or Musielak–Orlicz (MO) function will refer to a function

$$\varphi : \Omega \times [0, \infty) \to [0, \infty)$$

such that

$$\varphi(x, \cdot) : [0, \infty) \to [0, \infty)$$

is an Orlicz function for each fixed $x \in \Omega$ and

$$\varphi(\cdot, y) : \Omega \to [0, \infty)$$

is Lebesgue measurable for each fixed $y \in \mathbb{R}$.

The convex modular

$$\rho_\varphi(u) := \int_\Omega \varphi(x, |u(x)|) \, dx$$

gives rise to the Musielak–Orlicz space $L^\varphi(\Omega)$, which is defined as the real-vector space of all extended-real valued, Borel-measurable functions $u$ on $\Omega$ for which

$$\rho_\varphi(\lambda u) := \int_\Omega \varphi(x, |u(x)|\lambda) \, dx < \infty \text{ for some } \lambda > 0,$$

furnished with the norm

$$\|u\|_\varphi = \inf \left\{ \lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda}) \, dx \leq 1 \right\}.$$ 

It is well known [5,12] that $L^\varphi(\Omega)$ is a Banach space.
The Musielak–Orlicz Sobolev space $W^{1,\varphi}(\Omega)$ is the vector space of all functions in $L^\varphi(\Omega)$ whose distributional derivatives are in $L^\varphi(\Omega)$, furnished with the norm

$$
\|u\|_{1,\varphi} = \|u\|_\varphi + \|\nabla u\|_\varphi,
$$

where $\nabla \cdot$ stands for the gradient operator and $| \cdot |$ denotes the Euclidean norm in $\mathbb{R}^n$. It is well known (see [11]) that $W^{1,\varphi}(\Omega)$ is a Banach space under the assumptions

$$
\int_K \varphi(x,t) \, dx < \infty
$$

for any $K \subset \Omega$ with Lebesgue measure $|K| < \infty$ and

$$
\inf_{x \in \Omega} \varphi(x,1) > 0.
$$

The Sobolev space $W^{1,\varphi}_0(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

### 3 | SOBOLEV-TYPE EMBEDDINGS

Sobolev embeddings play an undeniable central role in analysis. In particular, the compactness of the embedding

$$
W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)
$$

is key in a variety of mathematical situations in which one needs to extract a convergent subsequence from a bounded sequence. The natural question arises whether a similar embedding theorem holds true in the general framework of the Musielak–Orlicz spaces. One can derive some encouragement from the validity of such result in the particular case of the variable exponent Lebesgue spaces under, as it will be shown here, very mild conditions on the exponent (see [5,8–10]).

The following examples, however, show that one can't expect Sobolev-type embeddings to hold even in very simple settings, without imposing some restrictions on the $MO$ function.

**Example 3.1.** We define the Musielak–Orlicz function

$$
\varphi : \Omega = (0, 1) \to [0, \infty)
$$

as

$$
\varphi(x,t) = \begin{cases} 
  t & \text{if } x \in \left[\frac{1}{n+1}, \frac{10n+1}{10n(n+1)}\right] \cup \left(\frac{10n+9}{10n(n+1)}, \frac{1}{n}\right], \\
  t^n & \text{if } x \in \left(\frac{10n+1}{10n(n+1)}, \frac{10n+9}{10n(n+1)}\right)
\end{cases}
$$

and consider the sequence $(u_n)$ defined by

$$
u_n(x) = \begin{cases} 
  1 & \text{for } x \in \left(\frac{10n+1}{10n(n+1)}, \frac{10n+9}{10n(n+1)}\right), \\
  10n(n+1) \left(x - \frac{1}{n+1}\right) & \text{if } x \in \left[\frac{1}{n+1}, \frac{10n+1}{10n(n+1)}\right], \\
  -10n(n+1) \left(x - \frac{1}{n}\right) & \text{if } x \in \left(\frac{10n+9}{10n(n+1)}, \frac{1}{n}\right)
\end{cases}
$$

**Example 3.2.** Here we set

$$
\varphi : \Omega = (0, 1) \to [0, \infty)
$$
defined by

\[
\varphi(x, t) = \begin{cases} 
    t^2 & \text{if } x \in \left( \frac{1}{n+1} + \frac{1}{10n(n+1)}, \frac{1}{n} - \frac{1}{10n(n+1)} \right), \\
    \frac{1}{n^2} t^2 & \text{if } x \in \left[ \frac{1}{n+1}, \frac{10n+1}{10n(n+1)} \right] \cup \left( \frac{10n+9}{10n(n+1)}, \frac{1}{n} \right).
\end{cases}
\]

In this case we define the sequence

\[
u_n(x) = \begin{cases} 
    n & \text{for } x \in \left( \frac{10n+1}{10n(n+1)}, \frac{10n+9}{10n(n+1)} \right), \\
    10n^2(n+1) \left( x - \frac{1}{n+1} \right) & \text{if } x \in \left[ \frac{1}{n+1}, \frac{10n+1}{10n(n+1)} \right), \\
    -10n^2(n+1) \left( x - \frac{1}{n} \right) & \text{if } x \in \left( \frac{10n+9}{10n(n+1)}, \frac{1}{n} \right).
\end{cases}
\]

In both examples above, no subsequence of \((u_n)\) is a Cauchy sequence in \(L^\varphi(0, 1)\), which implies that no subsequence can converge in the latter space. Yet, it is readily verified that both sequences are bounded in \(W^{1,\varphi}_0((0, 1))\).

## 4 Embeddings and the Matuszewska–Orlicz Index

We now set about to lay the foundations for our main result. We refer the reader to [5,12] for the detailed proof of the following theorem:

**Theorem 4.1.** Let \(\Omega \subset \mathbb{R}^n\) be bounded and let \(\theta\) and \(\phi\) be MO functions on \(\Omega\). The embedding

\[L^\theta(\Omega) \hookrightarrow L^\phi(\Omega)\]

is continuous if and only if there exist a constant \(c > 0\) and a function \(h \in L^1(\Omega)\) with \(\|h\|_1 \leq 1\) such that

\[\phi(x, tc) \leq \theta(x, t) + h(x);\]

and in this case, the norm of the embedding is comparable to \(\frac{1}{c}\).

The following embedding theorem is a fundamental result in analysis (see [1]):

**Theorem 4.2.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain, let \(p \in (1, n)\). Then the space \(W^{1,p}_0(\Omega)\) is compactly embedded in \(L^q(\Omega)\) for any \(q \in (1, \frac{np}{n-p})\). If \(p > n\), \(W^{1,p}_0(\Omega)\) embeds compactly in \(C(\overline{\Omega})\) (and hence in \(L^q(\Omega)\) for any \(q \in (1, \infty)\)). Finally, \(W^{1,p}_0(\Omega)\) is compactly embedded in \(L^q(\Omega)\) for any \(q \in [1, \infty)\).

We next introduce a generalized version of the Matuszewska–Orlicz index, which is to play a fundamental role in our further developments. The Matuszewska–Orlicz index of an Orlicz function \(\varphi\) was introduced by Matuszewska and Orlicz in [11].

**Definition 4.3.** For \(\varphi\) as above and each \(x \in \Omega\), set

\[M(x, t) = \lim_{u \to \infty} \varphi(x, tu) \varphi(x, u)^{-1}.
\]

The Matuszewska–Orlicz index of \(\varphi\) is defined to be

\[m(x) = \lim_{t \to \infty} \ln M(x, t) / \ln t = \inf_{t > 1} \ln M(x, t) / \ln t.
\]

**Definition 4.4.** The limit (4.1) is said to be **uniform** if for each \(\kappa > 0\) there exist \(s_0 > 1\) and \(T > 1\) such that, for all \((x, t) \in \Omega \times [T, \infty)\) and \(s \geq s_0\) one has

\[M(x, t) - \kappa < \frac{\varphi(x, ts)}{\varphi(x, s)} < M(x, t) + \kappa.
\]
We highlight the fact that in Example 2.1 the Matuszewska–Orlicz index of \( \varphi \) is unbounded on \( \Omega \), whereas in Example 2.2 it is equal to 2 on \( \Omega \). As we shall see, these observations are by no means incidental, for the behavior of the Matuszewska–Orlicz index in fact determines the validity of the Sobolev embedding (1.1).

The following examples illustrate the intrinsic meaning of above definition for some well known \( MO \) functions:

**Example 4.5.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain and let

\[
p : \Omega \to (0, \infty)
\]

be Borel-measurable. The \( MO \) function

\[
\varphi : \Omega \times [0, \infty) \to [0, \infty), \quad \varphi(x, t) = t^{p(x)}
\]

has Matuszewska index equal to \( p(x) \). In this case, the convergence (4.1) is trivially uniform on \( \Omega \) and the limit (4.2) is clearly uniform.

**Example 4.6.** Slightly less trivial is the uniform convergence of (4.1) for the \( MO \) function

\[
\varphi(x, t) = t^{p(x)}(1 + \log t)^{q(x)}
\]

where \( p \) is as in Example 4.5 and

\[
q : \Omega \to \mathbb{R}
\]

is a Borel-measurable, bounded function on \( \Omega \).

**Lemma 4.7.** If \( \varphi \) is an \( MO \) function for which the limits (4.1) and (4.2) are uniform, then there are constants \( C_1 > 1, C_2 > 1 \) and \( S_0 > 1 \) for which

\[
\varphi(x, C_1 s) \leq C_2 \varphi(x, s) \text{ for } s \geq S_0.
\]

**Proof.** A straightforward calculation shows that if \( \delta > 0 \) then there exists a constant \( C_1 > 1 \) for which \( t \geq C_1 \) implies

\[
M(x, t) < t^{m(x)+\delta};
\]

the assumed uniformity of the limit yields the existence of \( S_0 > 1 \) for which

\[
\sup_{s \geq S_0} \frac{\varphi(x, ts)}{\varphi(x, s)} < t^{m(x)+\delta} + \frac{1}{2} C_1 \sup_{s \leq S_0} \left[ \sup_{s \in [0, t]} m(x)+\delta \right]
\]

whenever \( s \geq S_0, t \geq C_1 \); in particular, setting \( t = C_1 \) in (4.3) one easily sees that for \( s \geq S_0 \) it holds that

\[
\varphi(x, C_1 s) \leq \frac{3}{2} C_1 \sup_{s \leq C_1} \left[ \sup_{s \in [0, t]} m(x)+\delta \right] \varphi(x, s),
\]

whence the lemma follows immediately.

\[\square\]

5 | THE COMPACTNESS OF THE EMBEDDING \( W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega) \)

We next move on to the main result.

**Theorem 5.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain and let

\[
\varphi : \Omega \times [0, \infty) \to \mathbb{R}
\]
be a locally integrable Musielak–Orlicz function (i.e., condition (2.2) holds) and such that the limits in (4.1) and (4.2) are uniform; assume that the Matuszewska index $m$ is the restriction to $\Omega$ of a continuous function $\bar{m}$ defined on the closure of $\Omega$, that

$$1 < m := \inf_{\Omega} m$$

and that there exists a function

$$\beta : (0, \infty) \rightarrow (0, \infty)$$

such that the inequality

$$\varphi(x, t) \leq \beta(t)$$

(5.1)

holds uniformly in $\Omega$. Then the embedding

$$W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega)$$

(5.2)

is compact.

Proof. Assume first $m_- < n$. It is first observed that if

$$0 < \gamma < \frac{m_-(n - m_-)}{2n - m_-}$$

then one has the inequality:

$$\frac{m_- - \gamma}{n - m_- + \gamma} > \frac{1}{2} \frac{m_-}{n - m_-} = \mu.$$ 

Let $r > 0$ be small enough for the inequality

$$\frac{n(n - r)}{r} > n + 2r$$

(5.3)

to hold. Let

$$\epsilon < \min\{\mu, 5r\}, \quad 0 < \gamma < \min\left\{\frac{\epsilon}{30}, \frac{m_-(n - m_-)}{2n - m_-}, \frac{m_- - 1}{2}\right\}.$$ 

(5.4)

Then, obviously

$$w(x) := m(x) - \gamma < m(x) < m(x) - \gamma + \frac{\epsilon}{20} = w(x) + \frac{\epsilon}{20}.$$ 

The uniformity conditions assumed on the index $m$, thus yield the existence of $\gamma$ satisfying (5.4) and of a constant $T_0 > 1$ for which $t \geq T_0$ implies, uniformly for all $t \geq T_0$, $x \in \Omega$:

$$t^{m(x) - \gamma} = t^{\mu(x)} < M(x, t) < t^{m(x) - \gamma + \epsilon/20} = t^{\mu(x) + \epsilon/20}.$$ 

The uniformity of the infimum (4.1) with respect to $t$ and $x$ yields a positive number $S_0$ (without loss of generality it can be assumed $S_0 > 1$) such that, for all $(x, t) \in \Omega \times [0, \infty)$, one has, for $t \geq T_0 > 1$ and any $\delta$ such that $\gamma < \delta < \frac{\epsilon}{20}$:

$$M(x, t) - \frac{1}{2} T_0^{m_- - \delta} < \frac{\varphi(x, tS_0)}{\varphi(x, S_0)} < M(x, t) + \frac{1}{2} T_0^{m_- - \delta}.$$ 

In all, the inequalities

$$\frac{1}{2} T_0^{\mu(x)} < \frac{\varphi(x, tS_0)}{\varphi(x, S_0)} < \frac{3}{2} T_0^{\mu(x) + \epsilon/20}.$$
hold uniformly in $\Omega$ for any $t \geq T_0$. Setting $tS_0 = s$, it is readily seen that for $s \geq T_0S_0$, one has

$$\frac{1}{2} \phi(x, S_0) \left( \frac{s}{S_0} \right)^{\psi(x)} \leq \phi(x, s) \leq \frac{3}{2} \phi(x, S_0) \left( \frac{s}{S_0} \right)^{\psi(x) + \frac{\epsilon}{50}}.$$  

(5.5)

Furthermore, assumptions (2.2) and (5.1) in conjunction with the assumptions on $w$ yield positive constants $A, B$ such that for all $x \in \Omega$ one has

$$\left( \phi(x, S_0) \right) \frac{1}{w(x)} \leq \sup_{\Omega} \left( \phi(x, S_0) \right) \frac{1}{w(x)} \leq B$$

and

$$A \leq \inf_{\Omega} \left( \phi(x, S_0) \right) \frac{1}{w(x)} \leq \left( \phi(x, S_0) \right) \frac{1}{w(x)}.$$  

Consequently, (5.5) implies the existence of positive constants $c_1, c_2$ for which

$$c_1 s w(x) < \phi(x, s) < c_2 s w(x) + \frac{\epsilon}{20},$$  

(5.6)

valid for all $x \in \Omega$ and $s \geq S_0 > 1$. By assumption and by virtue of Tietze’s extension theorem, $w$ is the restriction to $\Omega$ of a continuous function $p : \mathbb{R}^n \rightarrow [w_-, w_+]$. Let $p_1 = w_-$. For $k > 1$ if $p_{k-1} < n$, set

$$p_k = \frac{np_{k-1}}{n - p_{k-1}} - \frac{\epsilon}{5}.$$  

Clearly, if $p_{j-1} < n$, $p_j > p_1 + (j - 1)\frac{4}{5}\epsilon$ and $p_{j-1} < p_j$. Let $J$ be the first subindex for which $p_J > n - \frac{\epsilon}{2}$, where $r$ is as in (5.3). Let $I = [w_-, w_+]$ and

$$\Omega_1 = p^{-1}\left( \left( \frac{w_- + 1}{2}, \frac{np_1}{n - p_1} - \frac{\epsilon}{10} \right) \cap I \right)$$

and for $1 < k \leq J - 1$ set

$$\Omega_k = p^{-1}\left( \left( \frac{p_k}{n - p_k} - \frac{\epsilon}{10} \right) \cap I \right);$$

furthermore, define $\Omega_J$ and $\Omega_{J+1}$ as

$$\Omega_J := p^{-1}\left( \left( n - r, n + r \right) \cap I \right), \quad \Omega_{J+1} := p^{-1}\left( \left( n + \frac{r}{2}, \infty \right) \cap I \right),$$

and let $(\chi_k)_{1 \leq k \leq J+1}$ be a partition of unity subordinated to the cover $(\Omega_k)_k$ of $\Omega$. A straightforward argument shows that if $v \in C_0^\infty(\Omega)$ (which can be considered extended by 0 to $\mathbb{R}^n$) then, for each $k : 1 \leq k \leq J + 1$, $v \chi_k \in C_0^\infty(\Omega \cap \Omega_k)$. It follows from this observation that if $v \in W_0^{1,w}(\Omega)$, then for each fixed $k$,

$$v \chi_k \in W_0^{1,w}(\Omega \cap \Omega_k).$$

Fix a sequence $(u_j)_j$ bounded in $W_0^{1,w}(\Omega)$; then inequalities (5.6) in concert with a simple calculation imply that $(u_j)_j$ is bounded in $W_0^{1,w}(\Omega)$. We contend that $(u_j \chi_k)_j$ is bounded in $W_0^{1,p_k}(\Omega)$, for any subindex $k : 1 \leq k \leq J - 1$. Denote the indicator function of any set $A$ by $I_A$. Then by construction

$$w_k := w I_{\Omega \cap \Omega_k} \geq p_k I_{\Omega \cap \Omega_k}$$

and
so the embedding
\[
W^{1,20}_0(\Omega) \hookrightarrow W^{1,p_k,\Gamma_x,\Omega_k}_0(\Omega)
\]  
(5.7)
is continuous, that is, for some positive constant \( C \)
\[
\|u_j x_k\|_{W^{1,p_k,\Gamma_x,\Omega_k}_0(\Omega)} \leq C \|u_j x_k\|_{W^{1,20}_0(\Omega)}.
\]

On the other hand, if \( F_{kj} \) stands for any of the functions \( u_j x_k, (\nabla u_j) x_k \) or \( u_j \nabla x_k \)
\[
1 = \int_\Omega \frac{|F_{kj}|}{\|F_{kj}\|_{p_k}} = \int_\Omega \frac{|F_{kj}|}{\|F_{kj}\|_{p_k}} I_{\Omega \Delta \Omega_k}^{p_k}
\]
and with the same token
\[
1 = \int_\Omega \frac{|F_{kj}|}{\|F_{kj}\|_{w_k}} = \int_\Omega \frac{|F_{kj}|}{\|F_{kj}\|_{w_k}} I_{\Omega \Delta \Omega_k}^{w_k}
\]  
(5.8)
the two preceding observations and (5.7) yield
\[
\|F_{kj}\|_{p_k} = \|F_{kj}\|_{p_k} I_{\Omega \Delta \Omega_k} \leq C \|F_{kj}\|_{w_k} = \|F_{kj}\|_{w},
\]
which yields the contention.

Hence, by virtue of Theorem 4.2 there is no loss of generality in assuming that \( (u_j x_k) \) converges in \( L^{\frac{np_k}{20} - \frac{\epsilon}{50}}(\Omega) \). For simplicity, let \( q_k \) be the right-endpoint of \( \rho(\Omega_k) \) for \( 1 \leq k \leq J \). Next, if \( 1 \leq j \leq J \), set
\[
d_j := (q_j + \frac{\epsilon}{20}) I_{\Omega \Delta \Omega_j} + (w_+ + \frac{\epsilon}{20}) I_{\Omega \setminus \Omega_j}.
\]
Then \( d_j \geq w + \frac{\epsilon}{20} \) for all \( x \in \Omega \) and one has the continuous embedding
\[
L^{d_j}(\Omega) \hookrightarrow L^{w + \frac{\epsilon}{50}}(\Omega).
\]  
(5.9)
For any function \( u \in W^{1,\varphi}_0(\Omega) \) and \( 1 \leq k \leq J \):
\[
\int_\Omega |u x_k|^{d_j} = \int_\Omega |u x_k|^{d_j} I_{\Omega \Delta \Omega_k} = \int_\Omega |u x_k|^{q_k + \frac{\epsilon}{50}} = \left\{ \begin{array}{ll}
\int_\Omega |u x_k|^{\frac{np_k}{20} - \frac{\epsilon}{50}} & \text{if } k < J,
\int_\Omega |u x_k|^{w - \frac{\epsilon}{50}} & \text{if } k = J.
\end{array} \right.
\]
The preceding string of inequalities yields the following observation: If \( (u_j x_k) \) is a Cauchy sequence in \( L^{\frac{np_k}{20} - \frac{\epsilon}{50}}(\Omega) \), \( 1 \leq k \leq J - 1 \), then it is convergent in \( L^{d_j}(\Omega) \) and by virtue of (5.9), \( (u_j x_k) \) converges in \( L^{w + \frac{\epsilon}{50}}(\Omega) \). We claim that the latter observation yields the convergence of \( (u_j x_k) \) in \( L^{\varphi}(\Omega) \) for \( 1 \leq k \leq J - 1 \). Indeed, there is no loss of generality by assuming that \( (u_j x_k) \) converges pointwise a.e.; on the other hand:
\[
\int_\Omega \varphi(x, |u_j(x) - u_i(x)| x_k(x)) \, dx = \int_{\{|x|: |u_j(x) - u_i(x)| x_k(x) \leq S_0\}} \varphi(x, |u_j(x) - u_i(x)| x_k(x)) \, dx
\]
\[
+ \int_{\{|x|: |u_j(x) - u_i(x)| x_k(x) > S_0\}} \varphi(x, |u_j(x) - u_i(x)| x_k(x)) \, dx.
\]  
(5.10)
Since for any fixed \( x \in \Omega \) \( \varphi(x, \cdot) \) is nondecreasing, the integrand in the first term above satisfies the inequality
\[
\varphi(x, |u_j(x) - u_i(x)| x_k(x)) \leq \varphi(x, S_0).
\]
The assumption of local integrability on \( \varphi \) and a straightforward application of Lebesgue’s dominated convergence yield

\[
\lim_{i,j \to \infty} \int_{\{x : |u_j(x) - u_i(x)| \leq S_0 \}} \varphi(x, |u_j(x) - u_i(x)| \chi_k(x)) \, dx = 0.
\]

Since \( S_0 > 1 \), the second integral in (5.10) is dominated by

\[
\int_{\{x : |u_j(x) - u_i(x)| > S_0 \}} |u_j(x) - u_i(x)|^{1/2} \chi_k(x) \, dx.
\]

In all, \( (\rho_\varphi((u_j - u_i)\chi_k)) \to 0 \) as \( i, j \to \infty \). Next, we resort to Lemma 4.7 to observe that for \( C_1 \) as in its statement, one has:

\[
\rho_\varphi(C_1(u_i - u_j)\chi_k) = \int_{\{x : |u_j(x) - u_i(x)| \leq S_0 \}} \varphi(x, C_1|u_j - u_i| \chi_k) \, dx + \int_{\{x : |u_j(x) - u_i(x)| > S_0 \}} \varphi(x, C_1|u_j - u_i| \chi_k) \, dx;
\]

a straightforward application of Lebesgue’s dominated convergence theorem on the first integral and the consideration of Lemma 4.7 in the second one easily yield

\[
\rho_\varphi(C_1(u_i - u_j)\chi_k) \to 0 \quad \text{as} \quad i, j \to \infty.
\]

It follows automatically by induction that for any \( l \in \mathbb{N} \) one has

\[
\rho_\varphi(C^l_1(u_i - u_j)\chi_k) \to 0 \quad \text{as} \quad i, j \to \infty,
\]

and it is concluded from here that the sequence \( (u_j\chi_k)_j \) is Cauchy in \( L^\varphi(\Omega) \), as claimed. The remaining intervals in the covering are handled similarly: define

\[
w_j := wI_{\Omega \cap \Omega_j} \geq (n-r)I_{\Omega \cap \Omega_j} \]

the embedding

\[
W_0^{1,w_j}(\Omega) \hookrightarrow W_0^{1,n-r}(\Omega)
\]

is bounded. Retaining the notation of the above discussion, the sequence \( (u_j\chi_j)_j \) is bounded in \( W_0^{1,n-r}(\Omega) \); without loss of generality it can be considered convergent in \( L^{\frac{m+2r}{r}}(\Omega) \), which by the choice of \( r \) in (5.3) is continuously embedded in \( L^{n+2r}(\Omega) \). Finally, setting

\[
h = (n + 2r)I_{\Omega_{j+1}} + \left(w_+ + \frac{\varepsilon}{20}\right)I_{\Omega \cap \Omega_{j+1}}
\]

it is clear that \( L^h(\Omega) \) is continuously embedded in \( L^{\frac{m}{r}+\frac{2r}{3}}(\Omega) \). It follows immediately that \( (u_j\chi_{j+1})_j \) is Cauchy in the latter space. Theorem 4.1 ensures now that \( (u_j\chi_{j+1})_j \) is convergent in \( L^\varphi(\Omega) \). Finally, via the continuous embeddings

\[
W_0^{1,\varphi}(\Omega) \hookrightarrow W_0^{1,w}(\Omega) \hookrightarrow W_0^{1,n+\frac{x}{2},\Omega_{j+1}} \hookrightarrow W_0^{1,n+\frac{x}{2}}(\Omega)
\]

the boundedness of \( (u_j\chi_{j+1})_j \) in \( W_0^{1,\varphi}(\Omega) \) yields its boundedness in \( W_0^{1,n+\frac{x}{2},\Omega_{j+1}}(\Omega) \) and by way of Theorem 4.2 it is readily concluded that \( (u_j\chi_{j+1})_j \) can be considered convergent in \( C(\overline{\Omega}) \), hence convergent in \( L^\varphi(\Omega) \).

In all, for \( m_- < n \) any bounded sequence \( (u_j)_j \subset W_0^{1,\varphi}(\Omega) \) has a subsequence that converges in \( L^\varphi(\Omega) \). Since the case \( n \leq m_- \) is handled along the same lines, we only sketch the proof for this instance.

For \( r \) chosen as in (5.3) we observe that there exists \( T_0 > 1 \) such that uniformly on \( \Omega \) and for all \( t \geq T_0 \):

\[
t^{m(x)-\frac{r}{2}} < M(x,t) < t^{m(x)+r}.
\]
As before, one can conclude that given the conditions on the index, there are positive constants \( c_1 > 1, c_2 > 1 \) and \( T > 1 \) for which

\[
c_1 t^{n-r} \leq \varphi(x,t) < t^{m(x)+r}
\]  

(5.11)

uniformly in \( \Omega \), for all \( t \geq T \). Consider a partition of unity \((\chi_1, \chi_2)\) subordinated to the cover of \( \Omega \) that consists of the open sets

\[
\Omega_1 = p^{-1}((n-r, n+r) \cap I), \quad \Omega_2 = p^{-1}\left((n + \frac{r}{2}, \infty) \cap I\right).
\]

If \((u_j)\) is a bounded sequence in \( W_0^{1,\varphi}(\Omega) \) (hence in \( W_0^{1,n-r}(\Omega) \)) one can set

\[
q = \left(n - \frac{r}{4}\right) I_{\Omega_1} + m_\Omega I_{\Omega_2 \setminus \Omega_1}
\]

and along the same lines as (5.7)–(5.8) conclude that \((u_j \chi_1)\) is bounded in \( W_0^{1,n-r}(\Omega) \). Via Theorem 4.2 and the choice (5.3) it follows that \((u_j \chi_1)\) has a subsequence that converges in \( L^{n+2r}(\Omega) \). If

\[
t := (n + 2r) I_{\Omega \cap \Omega_1} + (m_\Omega + 2r) I_{\Omega \setminus \Omega_1},
\]

then the obvious equality

\[
\int_\Omega |(u_i - u_j) \chi_1|^{n+2r} = \int_\Omega |(u_i - u_j) \chi_1|^t
\]

implies that the subsequence also converges in \( L^t(\Omega) \) and since \( m + r < t \) in \( \Omega \) it converges also in \( L^{m+r}(\Omega) \), hence in \( L^\varphi(\Omega) \) via the right-hand inequality in (5.11). Still denoting this subsequence by \((u_j \chi_1)\), it is easy to see, that \((u_j \chi_2)\) is bounded in \( W_0^{1,n-r}(\Omega) \); therefore from Theorem 4.2 it is clear that it has a subsequence (still denoted by \((u_j \chi_2)\)) that converges in \( L^{m+2r}(\Omega) \). The right-hand inequality in (5.5) yields the convergence of \((u_j \chi_2)\) in \( L^\varphi(\Omega) \).

A straightforward computation reveals that the above conclusion implies the compactness of the embedding (5.2) in all cases.

It is apparent from the proof of the preceding theorem functions in \( W_0^{1,\varphi}(\Omega) \) belong to a higher-order integrability space than just \( L^\varphi(\Omega) \): We state this important fact as a separate corollary:

**Corollary 5.2.** For an \( \text{MO} \) function \( \varphi \) satisfying the conditions of Theorem 5.1, the embedding

\[
W_0^{1,\varphi}(\Omega) \hookrightarrow L^{m(x)+\frac{r}{m}}(\Omega) \subset L^\varphi(\Omega)
\]

is compact.

**Corollary 5.3.** For \( \varphi \) satisfying the conditions of Theorem 5.1, there exists a positive constant \( C \) depending only on \( n, \Omega, \varphi \) such that for any \( u \in W_0^{1,\varphi}(\Omega) \)

\[
\|u\|_{\text{var}} \leq C \|\nabla u\|_{\varphi}.
\]

**Proof.** If not, it would be an elementary matter to construct a sequence

\[(v_k)_k \subset W_0^{1,\varphi}(\Omega)\]

with

\[
\|v_k\|_{1,\varphi} = 1 \geq \|v_k\|_{\varphi} \geq k \|\nabla v_k\|_{\varphi} \quad \text{for} \quad k \in \mathbb{N}.
\]

Clearly,

\[
\|v_k\|_{1,\varphi} \to 0 \quad \text{in} \quad L^\varphi(\Omega)
\]

(5.12)
as \( k \to \infty \) and the compactness of the Sobolev embedding yields the existence of \( v \in L^p(\Omega) \) for which

\[
v_k \rightharpoonup v \quad \text{in} \quad L^p(\Omega).
\]

Necessarily then,

\[
\|v_k - v_j\|_{L^p(\Omega)} = \|v_k - v_j\|_p + \|\nabla (v_k - v_j)\|_p \to 0 \quad \text{as} \quad k, j \to \infty;
\]

it follows that \((v_k)_k\) converges in \(W^{1,p}_0(\Omega)\) and it is obvious that the limit must be \( v \). On the other hand, (5.12) forces \( \nabla v = 0 \) and hence \( v = 0 \), which is a contradiction.

\[\square\]

Remark 5.4. A quick examination of the counterexamples presented before Theorem 5.1 reveals that in Example 3.1, the Matuszewska index \( p \) (being unbounded on \((0,1)\)) is not the restriction to \( \Omega = (0,1) \) of any continuous function on \( \mathbb{R} \), whereas in Example 3.2 the estimate (5.1) is violated.

In what follows we present an example to the effect that the uniform convergence in the limits (4.1) and (4.2) cannot be weakened.

**Example 5.5.** Let \( \Omega \) stand for the Euclidean unit ball in \( \mathbb{R}^6 \) and for \( n \in \mathbb{N} \), let \( B_n \) be the ball of radius \( 2^{-n-2} \) centered at \( x_n = (2^{-n}, 0, 0, 0, 0, 0) \); denote by \( B_n, - \) the ball concentric with \( B_n \) of radius \( 2^{-n-3} \) and set

\[
B_{n,+} = B_n \setminus B_{n,-}.
\]

For each \( h > 0 \), let the function \( v_h : \mathbb{R}^6 \to \mathbb{R} \) be defined as

\[
v_h(x) = \begin{cases} 
1 & \text{if } |x| \leq h, \\
2 - \frac{|x|}{h} & \text{if } |x| \in (h, 2h), \\
0 & \text{if } |x| \in (2h, \infty).
\end{cases}
\]

The sequence \((u_n)_n\) is defined then as

\[
u_n(x) = 2^{2(n+1)}(w_6)^{-\frac{1}{3}} v_{2^{-(n+1)}}(x - x_n),
\]

where \( w_6 \) is the Lebesgue measure of the unit ball in \( \mathbb{R}^6 \).

For each \( z \in (0, \infty) \), let

\[
s^3 - z^2 = 3s^2(s - z)
\]

and consider the \( MO \) function

\[
\phi_z(t) = \begin{cases} 
t^2 & \text{if } t \in (0, z), \\
3s^2(s - z) + z^2 & \text{if } t \in (z, s), \\
s^3 & \text{if } t \in (s, \infty).
\end{cases}
\]

For each \( x \in \Omega \) choose \( z_n \approx |\nabla u_n(x)| \)

\[
\varphi(x, t) = \begin{cases} 
t^3 & \text{if } x \in \Omega \setminus \bigcup_n B_n, \\
\phi_{z_n}(t) & \text{if } x \in B_{n,+}.
\end{cases}
\]

It is easily checked that there are positive constants \( k_1, k_2 \) such that for all \( n \in \mathbb{N} \), one has:

\[
k_1 \leq \|u_n\|_\varphi \leq k_2, \quad k_1 \leq \|\nabla u_n\|_\varphi \leq k_2.
\]
and that no subsequence of \((u_n)\) converges in \(L^\varphi(\Omega)\). Notice that here the Matuszewska index is equal to 3 in \(\Omega \setminus \{0\}\) and that
\[
\varphi(x, t) \to t^2 \quad \text{as} \quad x \to 0 \quad \text{on} \quad B_{n,+},
\]
\[
\varphi(x, t) \to t^3 \quad \text{as} \quad x \to 0 \quad \text{on} \quad \Omega \setminus \bigcup B_{n,+}.
\]

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In particular, for the Sobolev Musielak–Orlicz spaces associated to the following MO functions, the embedding (1.1) is compact, for the following \(MO\) functions

\[
\left(1 < p_- =: \inf_{x \in \Omega} p(x) \leq p_+ =: \sup_{x \in \Omega} p(x) < \infty\right) :
\]

- \(\varphi(x, t) = t^{p(x)}(\log(1 + t))^{q(x)}\) for \(p, q \in C(\overline{\Omega})\), \(1 < p_- \leq p_+ < \infty\).
- \(\varphi(x, t) = t^{p(x)}|1 + |x||^{-n}, p \in C(\overline{\Omega})\).
- \(\varphi(x, t) = \frac{t^{p(x)}}{p(x)}\), \(p \in C(\overline{\Omega})\).

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