Dunkl transforms and Dunkl convolutions on functions and distributions with restricted growth

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In this paper we investigate Dunkl transforms and Dunkl convolutions on $\mathbb{R}$ in some spaces of functions and distributions with exponential growth introduced by Hasumi [12].

1 Introduction

Dunkl operators are differential-difference operators on $\mathbb{R}^n$ related to finite reflection groups. These operators can be seen as generalizations of partial derivatives and they play an important role in the description of Calogero–Moser–Sutherland models of quantum many-body systems on the line. Then the development of a distributional theory in the Dunkl setting can be very useful. The main parts of the Dunkl theory were analyzed in [6]–[8], [13], [17], [20], [21] and [24]. Among the various variants of Dunkl operators, the rational ones allow us to investigate harmonic analysis in close analogy with the classical Fourier analysis on $\mathbb{R}^n$.

In this paper we consider a special case of Dunkl operators: the rank-one case on $\mathbb{R}$, where the reflection group on $\mathbb{R}$ is $\mathbb{Z}_2$. The Dunkl operator can be written as

$$D_\mu f(x) = Df(x) + \frac{\mu}{x}(f(x) - f(-x)),$$

where $D = \frac{d}{dx}$ and $\mu \geq 0$. In [19] Rosenblum presents the main elements of harmonic analysis associated with the operator $D_\mu$. The function $e_\mu$ defined by

$$e_\mu(x) = \sum_{m=0}^{\infty} \frac{x^m}{\gamma_\mu(m)}, \quad x \in \mathbb{C},$$

where

$$\gamma_\mu(2m) = 2^{2m}m! \left(\mu + \frac{1}{2}\right)_m \quad \text{and} \quad \gamma_\mu(2m+1) = 2^{2m+1}m! \left(\mu + \frac{1}{2}\right)_{m+1}, \quad m \in \mathbb{N},$$

plays the role of a generalized exponential function. Here $(x)_m, x \geq 0, m \in \mathbb{N}$, represents the Pochhammer symbol, that is, $(x)_m = \Gamma(x+m)/\Gamma(x)$. Note that when $\mu = 0$, $e_\mu$ reduces to the usual exponential function.

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Next formula ([19, (2.5.5)]) relates $D_{\mu}$ and $e_{\mu}$,
\begin{equation}
D_{\mu} e_{\mu}(\lambda x) = \lambda e_{\mu}(\lambda x), \quad \lambda, x \in \mathbb{C}. 
\end{equation}

By taking the function $e_{\mu}$ as kernel the following generalized Fourier transform $\mathcal{F}_\mu$ is defined
\begin{equation}
\mathcal{F}_\mu(f)(x) = \int_{-\infty}^{+\infty} e_{\mu}(-ixy) f(y) \frac{|y|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} dy, \quad x \in \mathbb{R},
\end{equation}
where $f$ is a suitable function. This transform was investigated, in a more general setting, by de Jeu [13]. In [8, Corollary 2.7] a Plancherel theorem for $\mathcal{F}_\mu$ was established. Moreover, $\mathcal{F}_\mu$ is an automorphism on the Schwartz space $\mathcal{S}$ and the inverse of $\mathcal{F}_\mu$ is given on $\mathcal{S}$ by
\begin{equation}
\mathcal{F}_\mu^{-1}(g)(y) = \int_{-\infty}^{+\infty} e_{\mu}(ixy) g(x) \frac{|x|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} dx, \quad y \in \mathbb{R},
\end{equation}
([13, Corollary 4.22]). The integral transform $\mathcal{F}_\mu$, that reduces to Fourier transform when $\mu = 0$, appears in the physics literature on Bose-like oscillators ([16, p. 294]).

It is remarkable that the function $e_{\mu}(-ix)$ can be written as ([19, (3.1.2.])
\begin{equation}
e_{\mu}(-ix) = \Gamma(\mu + 1/2) 2^{\mu-1/2} |x|^{\mu+1/2} J_{\mu+1/2}(\sigma(x) J_{\mu+1/2}(|x|)), \quad x \in \mathbb{R},
\end{equation}
where $J_{\nu}$ represents the Bessel function of the first kind and order $\nu$. From (1.2) it follows that $\mathcal{F}_\mu$ reduces to a Hankel transform when it acts on even functions. This property also holds for the Dunkl transform on $\mathbb{R}^n$ acting on radial functions ([23, Proposition 1.4.8]).

By using (1.1) we obtain
\begin{equation}
D_{\mu} \mathcal{F}_\mu f = \mathcal{F}_\mu(-iyf),
\end{equation}
and by (1.2) and integration by parts we get
\begin{equation}
\mathcal{F}_\mu(D_{\mu} f) = i x \mathcal{F}_\mu(f),
\end{equation}
for each $f \in \mathcal{S}$.

Dunkl convolution operators for the Dunkl transforms $\mathcal{F}_\mu$ were investigated in [22]. If $f, g \in L^1(\mathbb{R}, |y|^{2\mu} dy)$ the Dunkl convolution $f \#_{\mu} g$ is defined through
\begin{equation}
(f \#_{\mu} g)(x) = \int_{-\infty}^{+\infty} \mu_{x} f(y) g(-y) \frac{|y|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} dy, \quad x \in \mathbb{R},
\end{equation}
where the Dunkl translation operator $\mu_{x}, x \in \mathbb{R}$, is given by
\begin{equation}
\mu_{x} f(y) = \int_{-\infty}^{+\infty} f(z) dv_{\mu,x,y}(z), \quad y \in \mathbb{R}.
\end{equation}

Here, for every $x, y \in \mathbb{R}$, $dv_{\mu,x,y}$ is a signed measure defined as follows,
\begin{equation}
dv_{\mu,x,y}(z) = \begin{cases}
K_{\mu}(x, y, z) \frac{|z|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} dz, & x, y \in \mathbb{R} \setminus \{0\}, \\
\delta_{y}, & y \in \mathbb{R} \text{ and } x = 0, \\
\delta_{x}, & x \in \mathbb{R} \text{ and } y = 0,
\end{cases}
\end{equation}
where as usual $\delta_{x}$ represents the Dirac functional supported on $\{a\}$, and
\begin{equation}
K_{\mu}(x, y, z) = \left(1 - \sigma_{x,y,z} + \sigma_{x,z,y} + \sigma_{z,y,x} \right) \rho_{\mu}(|x|, |y|, |z|),
\end{equation}
where
\[
\sigma_{x,y,z} = \begin{cases} 
\frac{x^2 + y^2 - z^2}{2xy}, & x, y \in \mathbb{R} - \{0\}, \ z \in \mathbb{R}, \\
0, & \text{otherwise},
\end{cases}
\]
and
\[
\rho_\mu(a,b,c) = \begin{cases} 
\beta_\mu \frac{((a+b)^2 - c^2)(c^2 - (a-b)^2))^{\mu-1}}{(abc)^{2\mu-1}}, & c \in (|a-b|, a+b), \ a, b, c > 0, \\
0, & \text{otherwise},
\end{cases}
\]
where \(\beta_\mu = 2^{3/2-\mu} (\Gamma(\mu + 1/2))^2 / (\sqrt{\pi} \Gamma(\mu))\). Dunkl translation operators are not positive operators (\cite{23}).

Dunkl transforms, convolutions and translations are related by (\cite[pp. 444 and 445]{14})
\[
\mathcal{F}_\mu(\mu \tau_x f)(y) = e_{\mu}(ixy) \mathcal{F}_\mu(f)(y), \quad x, y \in \mathbb{R},
\]
and
\[
\mathcal{F}_\mu(f \#_\mu g)(y) = \mathcal{F}(f)(y) \mathcal{F}_\mu(g)(y), \quad y \in \mathbb{R},
\]
that hold for each \(f, g \in L^1(\mathbb{R}, |x|^{2\mu} dx)\).

Dunkl convolutions \(\#_\mu\) have been investigated recently in distribution spaces by Betancor \cite{2} and Mohamed and Trimèche \cite{14}.

Since, for every \(x \in \mathbb{R}\) the Dunkl translation \(\mu \tau_x\) defines a continuous linear mapping from the Schwartz space \(S\) into itself (\cite[Proposition 2.3]{2}), the Dunkl convolution \(T \#_\mu \phi\) of \(T \in \mathcal{S}'\), the dual space of \(S\), and \(\phi \in \mathcal{S}\) is defined by
\[
(T \#_\mu \phi)(x) = (T(y), \mu \tau_x \phi(-y)), \quad x \in \mathbb{R}.
\]

Schwartz \cite{26} described the subspace \(\theta_e^0\) of the elements of \(\mathcal{S}'\) that define usual convolution operators on \(\mathcal{S}\) as follows. For every \(m \in \mathbb{Z}\), \(\theta_{e,m}\) denotes the space that consists of all those smooth functions \(\phi\) on \(\mathbb{R}\) such that, for each \(k \in \mathbb{N}\),
\[
\lim_{|x| \to \infty} (1 + |x|)^m D^k \phi(x) = 0.
\]
\(\theta_{e,m}\) is endowed with the topology associated with the family \(\{ \eta_{k}^m \}_{k \in \mathbb{N}}\) of seminorms, where
\[
\eta_{k}^m(\phi) = \sup_{x \in \mathbb{R}} (1 + |x|)^m |D^k \phi(x)|, \quad \phi \in \theta_{e,m} \text{ and } k \in \mathbb{N}.
\]
The union space \(\theta_e = \bigcup_{m \in \mathbb{Z}} \theta_{e,m}\) is equipped with the locally convex inductive limit topology. \(\theta_e^0\) represents the dual space of \(\theta_e\).

In \cite{2} Betancor established that \(\theta_e^0\) is the subspace of \(\mathcal{S}'\) of the \(\mu\)-Dunkl convolution operators for every \(\mu > 0\). Mohamed and Trimèche \cite{14} investigated Dunkl convolutions on \(\theta_{e,m}^0\), the dual space of \(\theta_{e,m}\), for \(m < 0\).

Hasumi \cite{12} and Zielezny \cite{29} studied Fourier transform and usual convolution on function and distribution spaces of exponential growth. Our objective in this paper is to analyze Dunkl transforms and Dunkl convolutions in the following spaces introduced by Hasumi (\cite{12}). We say that a smooth function \(\phi\) on \(\mathbb{R}\) is in \(\mathcal{H}\) when, for every \(m, k \in \mathbb{N}\),
\[
\gamma_{m,k}(\phi) = \sup_{x \in \mathbb{R}} e^{m|x|} |D^k \phi(x)| < \infty.
\]
\(\mathcal{H}\) is endowed with the topology generated by the family \(\{ \gamma_{m,k} \}_{m, k \in \mathbb{N}}\) of seminorms. Thus \(\mathcal{H}\) is a Fréchet space. By \(A\) we denote the space that consists of all those entire functions \(\Phi\) such that, for every \(m, k \in \mathbb{N}\),
\[
\eta_{m,k}(\Phi) = \sup_{|\text{Im} z| \leq k} (1 + |z|)^m |\Phi(z)| < \infty.
\]

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$\mathcal{A}$ is a Fréchet space when it is equipped with the topology generated by the system $\{\gamma_{m,k}\}_{m,k \in \mathbb{N}}$ of norms.

This paper is organized as follows. In Section 2 we prove that, for every $\mu > 0$, the Dunkl transform $\mathcal{F}_\mu$ is an isomorphism from $\mathcal{H}$ onto $\mathcal{A}$. To see this property we use a procedure developed by Anker [1] in his investigations about Fourier transform on Lie groups. Also we remark the power of the method showing that Dunkl transforms define isomorphisms between well-known spaces ([10], [11], [18], [25] and [27]). In Section 3 Dunkl convolutions are analyzed on $\mathcal{H}'$, the dual space of $\mathcal{H}$. We characterize the distributions on $\mathcal{H}'$ that define Dunkl convolution operators on $\mathcal{H}$ as the space $\theta_0'((L^\infty, L_\infty))$ considered by Hasumi [12].

Throughout this paper by $C$ we denote a positive constant not necessarily the same in each occurrence. We consider always $\mu > 0$.

## 2 Dunkl transforms and Hasumi spaces

In this section we study the behavior of Dunkl transforms on the spaces introduced by Hasumi. As was mentioned in the introduction the space $\mathcal{H}$ consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that

$$\gamma_{m,k}(\phi) = \sup_{x \in \mathbb{R}} e^{m|x|} |D^k \phi(x)| < \infty, \quad m, k \in \mathbb{N}. $$

$\mathcal{H}$ is a Fréchet and nuclear space when it is endowed with the topology associated with the family $\{\gamma_{m,k}\}_{m,k \in \mathbb{N}}$ of seminorms. A smooth function $f$ on $\mathbb{R}$ is a pointwise multiplier of $\mathcal{H}$, written $f \in \mathcal{M}_\mathcal{H}$, if and only if, for every $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that

$$\sup_{x \in \mathbb{R}} e^{-m_k|x|} |D^k f(x)| < \infty.$$  

In the following proposition we obtain new families of seminorms defining the topology of $\mathcal{H}$.

**Proposition 2.1** We define, for every $m, k \in \mathbb{N}$,

$$\gamma^\mu_{m,k}(\phi) = \sup_{x \in \mathbb{R}} e^{m|x|} |D^k_{\mu} \phi(x)|, \quad \phi \in \mathcal{H}.  

Then the family of seminorms $\{\gamma^\mu_{m,k}\}_{m,k \in \mathbb{N}}$ generates the topology of $\mathcal{H}$.

**Proof.** Assume that $\phi \in \mathcal{H}$. Let $m, k \in \mathbb{N}$. Note that, by (1.4),

$$D^k_{\mu} \phi(x) = \mathcal{F}_{\mu}^{-1}((iy)^k \mathcal{F}_{\mu}(\phi))(y), \quad x \in \mathbb{R}. $$

Since $\mathcal{F}_\mu$ is an automorphism of the Schwartz space $\mathcal{S}$ ([13, Corollary 4.22]) and the polynomials are multipliers of $\mathcal{S}$, there exist $C > 0$ and $n \in \mathbb{N}$ such that

$$\sup_{|x| \leq 1} e^{m|x|} |D^k_{\mu} \phi(x)| \leq C \sup_{x \in \mathbb{R}} |D^k \phi(x)| \leq C \max_{0 \leq j \leq n} \sup_{x \in \mathbb{R}} (1 + x^2)^n |D^j \phi(x)| \leq C \max_{0 \leq j \leq n} \sup_{x \in \mathbb{R}} e^{n|\phi(x)|} \leq C \sup_{x \in \mathbb{R}} e^{n|x|} |D^j \phi(x)|.$$  

Also, according to [14, Lemma 2.1 (iii)], there exists $C > 0$ for which

$$\sup_{|x| \geq 1} e^{m|x|} |D^k_{\mu} \phi(x)| \leq C \sum_{j=1}^k \gamma_{m,j}(\phi). $$

Thus we prove that the topology generated by $\{\gamma_{m,k}\}_{m,k \in \mathbb{N}}$ on $\mathcal{H}$ is stronger than the one defined by $\{\gamma^\mu_{m,k}\}_{m,k \in \mathbb{N}}$.

On the other hand we can write

$$D^k \phi(x) = \int_{-\infty}^{+\infty} D^k_x(e_{\mu}(ixy)) \mathcal{F}_{\mu}(\phi)(y) \frac{|y|^{\mu}}{2\mu^{1/2}1^{\mu+1/2}} dy, \quad x \in \mathbb{R}. $$
Moreover, according to [19, (2.3.5)],
\[ D_k^* e_{\mu}(i x y) = \frac{\Gamma(\mu + 1/2)}{\sqrt{\pi} \Gamma(\mu)} (i y)^k \int_{-1}^{1} e^{i x y t} t^k (1 - t)^{\mu-1} (1 + t)^{\mu} \, dt, \quad x, y \in \mathbb{R}. \]

Then, it follows that
\[ |D_k^* e_{\mu}(i x y)| \leq C |y|^k, \quad x, y \in \mathbb{R}, \quad (2.1) \]
and we have, by choosing \( l \in \mathbb{N} \) such that \( l > 2 \mu + 1 \),
\[
|D^k \phi(x)| \leq C \int_{-\infty}^{+\infty} |y|^k |\mathcal{F}_\mu(\phi)(y)| |y|^{2\mu} \, dy \\
\leq C \int_{-\infty}^{+\infty} \frac{|y|^{2\mu}}{1 + |y|^l} \, dy \sup_{y \in \mathbb{R}} |y|^k (1 + |y|^l) |\mathcal{F}_\mu(\phi)(y)| \\
\leq C \sup_{y \in \mathbb{R}} \int_{-\infty}^{+\infty} e_{\mu}(-i y) (|D^k_\mu \phi(z)| + |D^k_\mu D^l_\mu \phi(z)|) |z|^{2\mu} \, dz \\
\leq C \left( \sup_{z \in \mathbb{R}} e^{\sum_{k=0}^{l} |z|^2} |D^k_\mu \phi(z)| + \sup_{z \in \mathbb{R}} e^{\sum_{k=0}^{l} |z|^{2\mu}} |D^k_\mu D^l_\mu \phi(z)| \right), \quad x \in \mathbb{R}.
\]

Hence, we conclude
\[
\sup_{|x| \leq 1} \left| e^{m|x|} |D^k \phi(x)| \right| \leq C \left( \sup_{x \in \mathbb{R}} e^{\sum_{k=0}^{l} |z|^2} |D^k_\mu \phi(z)| + \sup_{z \in \mathbb{R}} e^{\sum_{k=0}^{l} |z|^{2\mu}} |D^k_\mu D^l_\mu \phi(z)| \right). \quad (2.2)
\]

A straightforward manipulation allows us to write
\[
|D^k \phi(x)| \leq C \sum_{j=0}^{k} \left( \left| (D^j_\mu \phi)(x) \right| + \left| (D^j_\mu \phi)(-x) \right| \right), \quad |x| \geq 1.
\]

Then, it follows that
\[
\sup_{|x| \geq 1} e^{m|x|} |D^k \phi(x)| \leq C \sum_{j=0}^{k} \gamma_{m,j}^\mu (\phi). \quad (2.3)
\]

By combining now (2.2) and (2.3) we obtain that the topology generated by \( \{ \gamma_{m,k}^\mu \}_{m,k \in \mathbb{N}} \) on \( \mathcal{H} \) is stronger than the one defined by \( \{ \gamma_{m,k} \}_{m,k \in \mathbb{N}} \).

Thus the proof of the property is finished. \( \square \)

The dual of \( \mathcal{H} \) is denoted, as usual, by \( \mathcal{H}' \). If \( f \) is a Lebesgue measurable function on \( \mathbb{R} \) such that \( e^{-m|x|} f \in L^1(\mathbb{R}, |x|^{2\mu} \, dx) \), for some \( m \in \mathbb{N} \), then \( f \) defines an element of \( \mathcal{H}' \), that we continue denoting by \( f \), through
\[
\langle f, \phi \rangle = \int_{-\infty}^{+\infty} f(x) \phi(x) \frac{|x|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} \, dx, \quad \phi \in \mathcal{H}. \quad (2.4)
\]

Thus the space of multipliers \( \mathcal{M}_\mathcal{H} \) can be seen as a subspace of \( \mathcal{H}' \). In [12, Proposition 3] Hasumi established that if \( T \in \mathcal{H}' \) then there exist \( m \in \mathbb{N} \) and a bounded continuous function \( f \) on \( \mathbb{R} \) such that
\[
\langle T, \phi \rangle = \int_{-\infty}^{+\infty} e^{m|y|} f(y) D^m \phi(y) \, dy, \quad \phi \in \mathcal{H}.
\]

The following new representation for the elements of \( \mathcal{H}' \) will be useful in the sequel. It can be proved by taking into account the Hahn–Banach and Riesz representation theorems and by using standard procedures (see, for instance, [28, p. 259]).
Proposition 2.2 Let $T$ be a functional on $\mathcal{H}$. Then $T \in \mathcal{H}'$ if, and only if, there exist $m \in \mathbb{N}$ and complex measures $w_{n,k}, n, k \in \mathbb{N}, n, k \leq m$, on $\mathbb{R}$ for which

$$
\langle T, \phi \rangle = \sum_{n,k=0}^{m} \int_{-\infty}^{+\infty} e^{n|x|}D_{\mu}^{k}\phi(x) \, dw_{n,k}(x), \quad \phi \in \mathcal{H}.
$$

In [12] Hasumi also considered the space $\mathcal{A}$ consisting of all those entire functions $\Phi$ such that, for every $m, k \in \mathbb{N}$,

$$
\eta_{m,k}(\Phi) = \sup_{\text{Im } z \leq k} (1 + |z|)^{m}\Phi(z) < \infty.
$$

$\mathcal{A}$ is endowed with the topology associated with the family \{\eta_{m,k}\}_{m,k \in \mathbb{N}} of norms. Thus $\mathcal{A}$ is a Fréchet space. The space $\mathcal{M}_{\mathcal{A}}$ of pointwise multipliers of $\mathcal{A}$ consists of those entire functions $\Psi$ such that, for every $k \in \mathbb{N}$, there exists $m_{k} \in \mathbb{N}$ for which

$$
\sup_{\text{Im } z \leq k} (1 + |z|)^{-m_{k}}|\Psi(z)| < \infty.
$$

To prove this we can proceed as in the proof of [5, Proposition 2.3].

By $\mathcal{A}'$ we denote the dual space of $\mathcal{A}$. If $f$ is a Lebesgue measurable function on $\mathbb{R}$ such that $(1 + |x|)^{m}f(x) \in L^{1}(\mathbb{R}, |x|^{2\mu} \, dx)$, for some $m \in \mathbb{N}$, then $f$ defines an element of $\mathcal{A}'$, that we continue denoting by $f$, through

$$
\langle f, \Phi \rangle = \int_{-\infty}^{+\infty} f(x)\Phi(x) \frac{|x|^{2\mu}}{2^{\mu+1/2}\Gamma(\mu + 1/2)} \, dx, \quad \Phi \in \mathcal{A}. \quad (2.5)
$$

Thus $\mathcal{M}_{\mathcal{A}}$ can be identified with a subspace of $\mathcal{A}'$.

In [12, Proposition 4] it was established that the Fourier transform is an isomorphism from $\mathcal{H}$ onto $\mathcal{A}$. We now prove that the Dunkl transform $\mathcal{F}_{\mu}$ is also an isomorphism from $\mathcal{H}$ onto $\mathcal{A}$. Our proof of this result is different from the proof of the corresponding result of Hasumi for the Fourier transform. We are inspired by the ideas developed by Anker [1] in his investigations about Fourier transform for semisimple Lie groups on Schwartz type spaces.

Theorem 2.3 The Dunkl transform is an isomorphism from $\mathcal{H}$ onto $\mathcal{A}$.

Proof. First we show that $\mathcal{F}_{\mu}$ is a continuous mapping from $\mathcal{H}$ into $\mathcal{A}$. Let $\phi \in \mathcal{H}$. According to [19, (2.3.5)] we have that

$$
e_{\mu}(ixz) = \frac{\Gamma(\mu + 1/2)}{\sqrt{\pi\Gamma(\mu)}} \int_{-1}^{1} e^{ixzt}(1 - t)^{\mu-1}(1 + t)^{\mu} \, dt, \quad x \in \mathbb{R} \text{ and } z \in \mathbb{C}. \quad (2.6)
$$

Then, for every $k \in \mathbb{N}$,

$$
D_{x}^{k}e_{\mu}(ixz) = \frac{\Gamma(\mu + 1/2)}{\sqrt{\pi\Gamma(\mu)}} (ixz)^{k} \int_{-1}^{1} e^{ixzt} t^{k}(1 - t)^{\mu-1}(1 + t)^{\mu} \, dt, \quad x \in \mathbb{R} \text{ and } z \in \mathbb{C}, \quad (2.7)
$$

and

$$
|D_{x}^{k}e_{\mu}(ixz)| \leq C |x|^{k}e^{\text{Im } z|x|}, \quad x \in \mathbb{R} \text{ and } z \in \mathbb{C}. \quad (2.8)
$$

Hence, the function $\Phi = \mathcal{F}_{\mu}(\phi)$ is entire and, for every $m, k \in \mathbb{N}$, by using (1.4), since $\mathcal{H} \subset \mathcal{S}$, we get

$$
\sup_{\text{Im } z \leq k} (1 + |z|)^{m} |\Phi(z)| \leq C \sum_{j=0}^{m} \sup_{\text{Im } z \leq k} |x^{j}\mathcal{F}_{\mu}(\phi)(z)|
$$

$$
\leq C \sum_{j=0}^{m} \sup_{\text{Im } z \leq k} \int_{-\infty}^{+\infty} |D_{y}^{j}\phi(y)||e_{\mu}(-izy)||y|^{2\mu} \, dy.
$$
\[
\leq C \sum_{j=0}^{m} \sup_{|\text{Im} z| \leq k} \int_{-\infty}^{+\infty} |D^{1-m}_{\mu} \phi(y)| e^{\text{Im} z |y|} |y|^{2\mu} \, dy \\
\leq C \sum_{j=0}^{m} \gamma_{k+1,j} (\phi).
\]

Thus by Proposition 2.1 we conclude that \( F_{\mu} \) maps \( \mathcal{H} \) continuously into \( \mathcal{A} \).

Now, since \( \mathcal{H} \) and \( \mathcal{A} \) are Fréchet spaces, to see that \( F_{\mu} \) is an isomorphism from \( \mathcal{H} \) into \( \mathcal{A} \) it is sufficient to prove that \( F_{\mu} \) is a one to one mapping from \( \mathcal{H} \) onto \( \mathcal{A} \). Note that according to [13, Corollary 4.22] \( F_{\mu} \) is a one to one mapping from \( \mathcal{H} \) into \( \mathcal{A} \). Then to finish the proof we have to show that \( F_{\mu} \) maps \( \mathcal{H} \) onto \( \mathcal{A} \).

By using [19, (2.3.5)] and by interchanging the order of integration we obtain, for every \( \phi \in \mathcal{S} \),

\[
F_{\mu}(\phi)(x) = \frac{1}{2^{\mu+1/2} \sqrt{\pi}} \int_{-\infty}^{+\infty} \phi(y) |y|^{2\mu} \int_{-1}^{1} e^{-ixyt} (1 - t^2)^{\frac{\mu-1}{2}} (1 + t) \, dt \, dy \\
= \frac{1}{2^{\mu+1/2} \sqrt{\pi}} \int_{-\infty}^{+\infty} \phi(y) |y|^{2\mu} \int_{-\infty}^{y} e^{-ixu} \left( 1 - \left( \frac{u}{y} \right)^2 \right)^{\frac{\mu-1}{2}} \left( 1 + \frac{u}{y} \right) \, du \, dy \\
= \frac{1}{2^{\mu+1/2} \sqrt{\pi}} \int_{-\infty}^{+\infty} \phi(y) \int_{-y}^{y} e^{-ixu} (y^2 - u^2)^{\frac{\mu-1}{2}} (y + u) \, du \, dy \\
= F_{\mu} \circ V_{\mu}(\phi)(x), \quad x \in \mathbb{R},
\]

where \( F_{\mu} \) represents the euclidean Fourier transform given by

\[
F_{0}(\phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(y) e^{-ixy} \, dy, \quad x \in \mathbb{R},
\]

and \( V_{\mu} \) denotes the operator

\[
V_{\mu}(\phi)(x) = \frac{1}{2^{\mu} \Gamma(\mu)} \int_{|y|>|x|} \phi(y) (y^2 - x^2)^{\frac{\mu-1}{2}} (y + x) \, dy, \quad x \in \mathbb{R}.
\]

This operator \( V_{\mu} \) coincides with the operator \( \mu^{1/2} V \) that appears in [15] when our Dunkl case is considered.

For every \( a > 0 \) we define as in [15] the spaces \( D_{a}(\mathbb{R}) \) and \( H_{a} \). \( D_{a}(\mathbb{R}) \) consists of all those smooth functions on \( \mathbb{R} \) having support contained in \([ -a, a] \). This space is Fréchet when it is considered on it the topology induced by the Schwartz space \( \mathcal{S} \).

By \( H_{a} \) we denote the space of entire functions \( \Phi \) such that, for every \( m \in \mathbb{N} \),

\[
p_{m}(\Phi) = \sup_{z \in \mathbb{C}} (1 + |z|)^m |\Phi(z)| e^{-\alpha |\text{Im} z|} < \infty.
\]

\( H_{a} \) is equipped with the topology generated by the family \( \{p_{m}\}_{m \in \mathbb{N}} \) of norms. Thus \( H_{a} \) is a Fréchet space. The inductive limit spaces \( \cup_{a>0} D_{a}(\mathbb{R}) \) and \( \cup_{a>0} H_{a} \) are denoted by \( D(\mathbb{R}) \) and \( H \), respectively.

It is not hard to see that \( D(\mathbb{R}) \) is a dense subspace of \( \mathcal{H} \). Since the Fourier transform \( F_{0} \) is an isomorphism from \( \mathcal{H} \) onto \( \mathcal{A} \) ([12, Proposition 4]) and from \( D(\mathbb{R}) \) onto \( H \) ([9, Section 1.1, Chapter II]), \( H \) is a dense subspace of \( \mathcal{A} \).

According to [15, Theorem 3.8] the Dunkl transform \( F_{\mu} \) is an isomorphism from \( D_{a}(\mathbb{R}) \) onto \( H_{a} \), for every \( a > 0 \), and from \( D(\mathbb{R}) \) onto \( H \).

Our next objective is to prove that the inverse \( F_{\mu}^{-1} \) of the Dunkl transform is a continuous mapping from \( H \) onto \( D(\mathbb{R}) \), when \( H \) is endowed with the topology induced by \( \mathcal{A} \) and \( D(\mathbb{R}) \) is equipped with the topology induced by \( \mathcal{H} \).

Let \( h \in H \). We choose \( f, g \in D(\mathbb{R}) \) such that \( F_{\mu}(f) = h \) and \( F_{0}(g) = h \). Then \( V_{\mu}(f) = g \).

Let \( k, m \in \mathbb{N} \). According to (2.1), if \( a > 0 \), by taking \( l \in \mathbb{N} \) such that \( l > 2\mu + k + 1 \), we get

\[
\sup_{|x| < a} e^{-a |x|} |D^{k} f(x)| \leq Ce^{ma} \int_{-\infty}^{+\infty} |h(y)| |y|^{k+2\mu} \, dy \leq Ce^{ma} \sup_{y \in \mathbb{R}} (1 + |y|)^{l} |h(y)|.
\]
Here $C$ does not depend on $a$.

For every $n \in \mathbb{N}$, $n \geq 1$, we choose a smooth function $w_n$ on $\mathbb{R}$ such that $w_n(x) = 0$, $|x| \geq n + 1$, $w_n(x) = 1$, $|x| \leq n$ and such that, for each $l \in \mathbb{N}$ there exists $C_l > 0$ (that does not depend on $n$) for which

$$\left| D^l w_n(x) \right| \leq C_l, \quad x \in \mathbb{R}. $$

Let $n \in \mathbb{N}$, $n \geq 1$. We write $g_n = (1 - w_n)g$. Then $g = w_ng + g_n$. Also we consider $h_n = \mathcal{F}_0(g_n)$. According to [15, Theorem 3.8] there exists $f_n \in \mathcal{D}(\mathbb{R})$ such that $V_{\mu}(f_n) = g_n$. Note that $\supp (V_{\mu}(f - f_n)) = \supp (w_ng) \subset [-n - 1, n + 1]$. Then, by using again [15, Theorem 3.8], $\supp (f - f_n) \subset [-n - 1, n + 1]$. Hence $f(x) = f_n(x)$, $|x| \geq n + 1$. By proceeding as in (2.9) we get

$$\sup_{n+1<|x|\leq n+2} e^{m|x|} |D^k f(x)| = \sup_{n+1<|x|\leq n+2} e^{m|x|} |D^k f_n(x)| \leq C e^{m(n+2)} \sup_{x \in \mathbb{R}} (1 + |x|)^l |h_n(x)|,$$

where as in (2.9) $l \in \mathbb{N}$ is chosen such that $l > 2\mu + k + 1$. Here $C > 0$ and $l \in \mathbb{N}$ do not depend on $n$.

By [12, Proposition 4] there exist $r \in \mathbb{N}$ and $C > 0$ for which

$$\sup_{x \in \mathbb{R}} (1 + |x|)^l |h_n(x)| \leq C \sum_{j=0}^r \sup_{x \in \mathbb{R}} e^{r|x|} |D^j g_n(x)| \leq C \sum_{j=0}^r \sup_{|x| \geq n} e^{r|x|} |D^j g_n(x)|. \tag{2.10} $$

Then,

$$\sup_{n+1<|x|\leq n+2} e^{m|x|} |D^k f(x)| \leq C \sum_{j=0}^r \sup_{|x| \geq n} e^{(r+m)|x|} |D^j g_n(x)| \leq C \sum_{j=0}^r \sup_{x \in \mathbb{R}} e^{(r+m)|x|} |D^j g_n(x)|. $$

By taking into account the properties of the functions $w_n$, $n \in \mathbb{N}$, a straightforward manipulation leads to

$$\sup_{n+1<|x|\leq n+2} e^{m|x|} |D^k f(x)| \leq C \sum_{j=0}^r \sup_{x \in \mathbb{R}} e^{(r+m)|x|} |D^j g(x)|. $$

Hence, [12, Proposition 4] implies that, for certain $\alpha, \beta \in \mathbb{N}$,

$$\sup_{n+1<|x|\leq n+2} e^{m|x|} |D^k f(x)| \leq C \eta_{\alpha, \beta}(h). $$

Since $C$ does not depend on $n$ we conclude from above estimates and (2.9) that

$$\gamma_{m,k}(f) \leq C \eta_{\alpha, \beta}(h). $$

Thus we prove that $\mathcal{F}^{-1}_{\mu}$ maps continuously $H$ into $\mathcal{D}(\mathbb{R})$ considering $H$ and $\mathcal{D}(\mathbb{R})$ as subspaces of $A$ and $\mathcal{H}$, respectively.

We now show that $\mathcal{F}_{\mu}$ maps $\mathcal{H}$ onto $A$. Let $\Phi \in A$. Since $H$ is a dense subspace of $A$, there exists a sequence $(\Phi_n)_{n \in \mathbb{N}} \subset H$ such that $\Phi_n \to \Phi$, as $n \to \infty$, in $A$. Then $(\mathcal{F}^{-1}_{\mu}(\Phi_n))_{n \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ is a Cauchy sequence in $\mathcal{H}$. Hence, for a certain $\phi \in \mathcal{H}$, $\mathcal{F}^{-1}_{\mu}(\Phi_n) \to \phi$, as $n \to \infty$, in $\mathcal{H}$. Since, as we showed at the beginning of this proof, $\mathcal{F}_{\mu}$ is a continuous mapping from $\mathcal{H}$ into $A$, we conclude that $\Phi_n \to \mathcal{F}_{\mu}\phi$, as $n \to \infty$, in $A$, and then $\Phi = \mathcal{F}_{\mu}\phi$.

Thus the proof is finished.
The Dunkl transform is defined on the dual space $\mathcal{H}'$ of $\mathcal{H}$ as the transpose of the Dunkl transform on $\mathcal{H}$, that is, if $T \in \mathcal{H}'$ the Dunkl transform $F_{\mu}T$ of $T$ is the element of $\mathcal{A}'$ given by
\[
\langle F_{\mu}T, \Phi \rangle = \langle T, F_{\mu}^{-1}\Phi \rangle, \quad \Phi \in \mathcal{A}.
\] (2.11)

Then, the Dunkl transform $F_{\mu}$ is an isomorphism from $\mathcal{H}'$ onto $\mathcal{A}'$ when we consider on the dual spaces the strong or the weak$^*$ topologies.

**Remark 2.4** A careful study of the procedure used in the proof of Theorem 2.3 allows us to see that it is useful to analyze the Dunkl transform $F_{\mu}$ in other known function spaces. We now present some results that can be proved by proceeding as in the proof of Theorem 2.3.

### 2.1 Dunkl transform on the space $K_M$

Let $w$ denote a continuous increasing function defined on $[0, \infty)$ such that $w(0) = 0$ and $\lim_{x\to\infty} w(x) = \infty$. We define
\[
M(x) = \int_0^x w(t) \, dt, \quad x \in [0, \infty).
\]

Thus $M$ is an increasing, convex, continuous function on $(0, \infty)$ such that the following convexity inequality holds:
\[
M(x) + M(y) \leq M(x + y), \quad x, y \in [0, \infty).
\] (2.12)

Two functions $M_1$ and $M_2$ associated with $w_1$ and $w_2$, respectively, as above, are said to be dual in the sense of Young when $w_1(w_2(x)) = x$ and $w_2(w_1(x)) = x, x \in [0, \infty)$. Then we have that ([11, p. 19])
\[
xy \leq M_1(x) + M_2(y), \quad x, y \in [0, \infty).
\] (2.13)

Examples of functions that are dual in the sense of Young are the following
\[
M_1(x) = \frac{x^p}{p}, \quad M_2(x) = \frac{x^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1;
\]
\[
M_1(x) = e^x - x - 1, \quad M_2(x) = (x + 1) \log(x + 1) - x.
\]

Some properties of this class of functions can be encountered in [11].

Assume that $M$ is a function as above. The space $K_M$ consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that
\[
\Gamma_{m,k}^M(\phi) = \sup_{x \in \mathbb{R}} e^{M(m|x|)}|D^k\phi(x)| < \infty, \quad m, k \in \mathbb{N}.
\]

This space is Fréchet when it is endowed with the topology associated to the system $\{\Gamma_{m,k}^M\}_{m,k \in \mathbb{N}}$ of seminorms. The space $D(\mathbb{R})$ is a dense subspace of $K_M$.

By $K_M$ we denote the space of entire functions $\Phi$ such that
\[
\Lambda_{m,k}^M(\Phi) = \sup_{z \in \mathbb{C}} (1 + |z|)^{k}e^{-M(|\text{Im} \, z|m)}|\Phi(z)| < \infty, \quad m, k \in \mathbb{N}, \quad m \neq 0.
\]

$K_M$ is endowed with the topology generated by the family $\{\Lambda_{m,k}^M\}_{m,k \in \mathbb{N}}$ of norms. Thus $K_M$ is a Fréchet space.

In [18, Corollary 4.2] Paik established that the Fourier transform is an isomorphism from $K_M$ onto $K_N$ provided that $M$ and $N$ are dual in the Young sense. This result had been previously proved by Sampson and Zielezny ([25, Theorem 4]) when $M(x) = x^p/p, x \in [0, \infty)$, and $1 < p < \infty$.

By using the same procedure as the one employed in the proof of Theorem 2.3 and by taking into account (2.12) and (2.13) we can obtain the following.

**Theorem 2.5** Let $M$ and $N$ be Young’s dual functions. Then, the Dunkl transform $F_{\mu}$ is an isomorphism from $K_M$ onto $K_N$.

Note that Theorem 2.3 is not included in Theorem 2.5, because the function $M(x) = x, x \in [0, \infty)$, does not satisfy the above specified condition. It can also be observed that, for every $m, k \in \mathbb{N}$, the seminorm $\eta_{m,k}$ includes a supremum extended to the strip $\{ |\text{Im} \, z| \leq k \}$ while in the definition of $\Lambda_{m,k}^M$ the supremum is extended to the whole complex plane.
2.2 Dunkl transform on W- and S-type spaces

To analyze Cauchy problems associated with partial differential equations Gelfand and Shilov ([11, Chapter 1]) considered function spaces known as W-type spaces. Assume that as in the previous paragraph $M$ is a function defined on $[0, \infty)$ by

$$M(x) = \int_0^x w(t) \, dt, \quad x \in [0, \infty),$$

where $w$ is a continuous increasing function on $[0, \infty)$ such that $w(0) = 0$ and $\lim_{x \to \infty} w(x) = \infty$.

Let $a > 0$. The space $W_{M,a}$ consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that, for every $n, k \in \mathbb{N}, n \neq 0$,

$$p_{n,k}^{M,a}(\phi) = \sup_{x \in \mathbb{R}} e^{M(a(1-n|x|))} |D^k\phi(x)| < \infty.$$

$W_{M,a}$ is a Fréchet space when it is equipped with the topology associated to the family $\{p_{n,k}^{M,a}\}_{n,k \in \mathbb{N}, n \neq 0}$ of seminorms. Moreover $D(\mathbb{R})$ is a dense subspace of $W_{M,a}$. By $W_{M,a}$ we denote the space consisting of all entire functions $\Phi$ for which

$$q_{n,k}^{M,a}(\Phi) = \sup_{z \in \mathbb{C}} e^{-M(a(n+2|\text{Im} z|)) (1 + |z|^k)|\Phi(z)|} < \infty,$$

for each $n, k \in \mathbb{N}$. $W_{M,a}$ is endowed with the topology generated by the system $\{q_{n,k}^{M,a}\}_{n,k \in \mathbb{N}}$ of norms. Thus $W_{M,a}$ is a Fréchet space.

In [11, Theorem 3, p. 28] it was established that the Fourier transform is an isomorphism from $W_{M,a}$ onto $W_{N,1/a}$, provided that $M$ and $N$ are Young’s dual functions.

By proceeding as in the proof of Theorem 2.3 and by using some ideas introduced in the proof of [3, Theorems 2.2 and 2.3] we can prove the next result.

**Theorem 2.6** Let $a > 0$. Assume that $M$ and $N$ are Young’s dual functions. Then the Dunkl transform $F_\mu$ is an isomorphism from $W_{M,a}$ onto $W_{N,1/a}$.

In [10, Chapter 4] Gelfand and Shilov studied a class of function spaces that they call $S$-type spaces.

If $\alpha > 0$ and $A > 0$ the space $S_{\alpha,A}$ consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that, for every $\delta > 0$ and $m \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}, k \in \mathbb{N}} \frac{|x^k D^m \phi(x)|}{(A + \delta)^k} < \infty.$$

The space $S_{\alpha,A}$ coincides with the space $W_{M,a}$ when $a = A^{-1}e^{-\alpha}$ and $M(x) = \alpha x^{1/\alpha}, x \in [0, \infty)$, provided that $0 < \alpha < 1$ ([10, p. 172]).

If $\beta > 0$ and $B > 0$, by $S^{\beta,B}$ we denote the space that consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that, for every $\rho > 0$ and $k \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}, m \in \mathbb{N}} \frac{|x^k D^m \phi(x)|}{(B + \rho)^m} < \infty.$$

This space $S^{\beta,B}$ is equal to $W_{M,a}$ when $M(x) = (1 - \beta)x^{1/(1-\beta)}, x \in [0, \infty), 0 < \beta < 1$ and $a = Be^\beta$ ([10, p. 210]).

Then, the following result is a consequence of Theorem 2.6.

**Corollary 2.7** Let $0 < \alpha < 1$ and let $A > 0$. Then the Dunkl transform $F_\mu$ is an isomorphism from $S_{\alpha,A}$ onto $S^{\alpha,A}$.

It is remarkable that our procedure does not apply to the study of the behavior of the Dunkl transform on the spaces $W_{N,b}^{M,a}$ of Gelfand and Shilov ([11, Chapter 1]). We conjecture that if $M_1$ and $N_1$ are the Young’s dual functions of $M$ and $N$, respectively, and $a, b > 0$, then the Dunkl transform $F_\mu$ is an isomorphism from $W_{N,b}^{M,a}$ onto $W_{N_1,a/b}$, but in this moment we do not know how to prove this result.
2.3 Dunkl transforms on the space $\mathcal{H}_r$

In [27] Sznajder introduced the following generalization of the spaces $\mathcal{H}$ and $\mathcal{A}$.

Let $r > 0$, $\mathcal{H}_r$ denotes the space of all smooth functions $\phi$ on $\mathbb{R}$ such that, for any $m, k \in \mathbb{N}$,

$$\gamma^r_{m,k}(\phi) = \sup_{x \in \mathbb{R}} e^{r_m|x|} |D^k\phi(x)| < \infty,$$

where $0 < r_0 < r_1 < \ldots$ and $r_m \to r$, as $m \to \infty$. The topology of $\mathcal{H}_r$ is defined by the seminorms $\{\gamma^r_{m,k}\}_{m,k \in \mathbb{N}}$. Thus $\mathcal{H}_r$ is a Fréchet space and $\mathcal{D}(\mathbb{R})$ is a dense subspace of $\mathcal{H}_r$. Note that $\mathcal{H}_r$ does not depend on the sequence $\{r_m\}_{m \in \mathbb{N}}$ converging to $r$.

The space $\mathcal{A}_r$ consists of all those holomorphic functions $\Phi$ on the strip $\{||\text{Im} \ z|| < r\}$ such that, for every $m, k \in \mathbb{N}$,

$$\eta^r_{m,k}(\Phi) = \sup_{|\text{Im} \ z| \leq r_m} (1 + |z|)^k |\Phi(z)| < \infty.$$  

Here the sequence $\{r_m\}_{m \in \mathbb{N}}$ is as above. $\mathcal{A}_r$ is equipped with the topology associated with the family $\{\eta^r_{m,k}\}_{m,k \in \mathbb{N}}$ of norms. In [27, Theorem 1] it is proved that the Fourier transform is an isomorphism from $\mathcal{H}_r$ onto $\mathcal{A}_r$.

By taking into account that, for every $n \in \mathbb{N}$, the function $h_n$ in the proof of Theorem 2.3 is in the Schwartz space $\mathcal{S}$, the inequality (2.10) can be replaced, for every $\varepsilon > 0$, by

$$\sup_{x \in \mathbb{R}} (1 + |x|)^{\varepsilon} |h_n(x)| \leq C \sum_{j=0}^{r} \sup_{x \in \mathbb{R}} (1 + |x|)^{\varepsilon} |D^j g_n(x)|$$

$$\leq C_\varepsilon \sum_{j=0}^{r} \sup_{x \in \mathbb{R}} e^{\varepsilon |z|} |D^j g_n(x)|,$$

where $r \in \mathbb{N}$ does not depend on $\varepsilon > 0$, and $C_\varepsilon$ is a suitable positive constant. Then, by proceeding as in the proof of Theorem 2.3 we obtain the following.

**Theorem 2.8** Let $r > 0$. Then the Dunkl transform $\mathcal{F}_\mu$ is an isomorphism from $\mathcal{H}_r$ onto $\mathcal{A}_r$.

3 Dunkl convolution in the space $\mathcal{H}$ and its dual $\mathcal{H}'$

In this section we study the behavior of Dunkl convolution on the space $\mathcal{H}$ and its dual $\mathcal{H}'$.

According to [14, pp. 444 and 445] we have, for every $\phi, \psi \in \mathcal{H}$, that

$$\mathcal{F}_\mu(\phi \#_\mu \psi)(x) = \mathcal{F}_\mu(\phi)(x)\mathcal{F}_\mu(\psi)(x), \quad x \in \mathbb{R}.$$  

Then, by invoking Theorem 2.3, since $\mathcal{A}$ is contained in $\mathcal{M}_A$ we conclude that

$$\phi \#_\mu \psi = \mathcal{F}_\mu^{-1}(\mathcal{F}_\mu(\phi)\mathcal{F}_\mu(\psi)), \quad \phi, \psi \in \mathcal{H},$$

and that the convolution mapping $(\phi, \psi) \mapsto \phi \#_\mu \psi$ is bilinear and continuous from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{H}$.

Also, the duplication formula [14, pp. 444 and 445] implies, for each $\phi \in \mathcal{H}$,

$$\mathcal{F}_\mu(\mu \tau_x \phi)(y) = e_{\mu}(ixy)\mathcal{F}_\mu(\phi)(y), \quad x, y \in \mathbb{R}.$$  

From Theorem 2.3 since, for every $x \in \mathbb{R}$, the function $e_{\mu}(ixy) \in \mathcal{M}_A$, one deduces

$$\mu \tau_x \phi = \mathcal{F}_\mu^{-1}(e_{\mu}(ixy)\mathcal{F}_\mu(\phi)(y)), \quad \phi \in \mathcal{H} \text{ and } x \in \mathbb{R},$$

and that, for all $x \in \mathbb{R}$, the mapping $\phi \to \mu \tau_x \phi$ is continuous from $\mathcal{H}$ into itself.

Moreover the following useful property for Dunkl translations holds.

**Proposition 3.1** Let $\phi \in \mathcal{H}$. The (nonlinear) mapping $x \to \mu \tau_x \phi$ is continuous from $\mathbb{R}$ into $\mathcal{H}$.  

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Proof. Let \( x_0 \in \mathbb{R} \) and \( \phi \in \mathcal{H} \). According to (3.1) and Theorem 2.3, to see that the mapping \( F(x) = \mu \tau_x \phi \) is continuous in \( x_0 \) it is sufficient to show that

\[
\lim_{x \to x_0} e_{\mu}(ixz)F_{\mu}(\phi)(z) = e_{\mu}(ix_0z)F_{\mu}(\phi)(z),
\]

in the sense of convergence in \( \mathcal{A} \). Assume that \( m, k \in \mathbb{N} \) and \( \varepsilon > 0 \). By taking into account (2.8) we obtain

\[
|e_{\mu}(ixz)| \leq C e^{k|x|} \leq C e^{k|x|}, \quad x \in \mathbb{R} \text{ and } |\text{Im } z| \leq k.
\]

Hence, since \( F_{\mu}(\phi) \in \mathcal{A} \) (Theorem 2.3), there exists \( r_0 > 0 \) for which

\[
\sup_{|\text{Im } z| \leq k, |\text{Re } z| > r_0} (1 + |z|)^m |F_{\mu}(\phi)(z)| |e_{\mu}(ixz) - e_{\mu}(ix_0z)| < \varepsilon, \quad |x - x_0| < 1.
\]

Finally, the continuity of the function \( e_{\mu} \) implies that there exists \( 0 < \delta < 1 \) such that

\[
\sup_{|\text{Im } z| \leq k, |\text{Re } z| \leq r_0} (1 + |z|)^m |F_{\mu}(\phi)(z)| |e_{\mu}(ixz) - e_{\mu}(ix_0z)| < \varepsilon, \quad |x - x_0| < \delta.
\]

Thus the proof is complete. \( \square \)

If \( T \in \mathcal{H}' \) and \( \phi \in \mathcal{H} \) we define the convolution \( T \#_{\mu} \phi \) through

\[
(T \#_{\mu} \phi)(x) = \langle T(t), \mu \tau_x \phi(-t) \rangle, \quad \phi \in \mathcal{H}.
\]

Note that according to Proposition 3.1 the function \( T \#_{\mu} \phi \) is continuous on \( \mathbb{R} \), for every \( T \in \mathcal{H}' \) and \( \phi \in \mathcal{H} \). As the following example shows, if \( T \in \mathcal{H}' \) and \( \phi \in \mathcal{H} \), \( T \#_{\mu} \phi \) is not always in \( \mathcal{H} \). Indeed, we consider the functional \( T \) defined on \( \mathcal{H} \) by

\[
\langle T, \phi \rangle = \int_{-\infty}^{+\infty} \phi(y) y^{2\mu} \, dy, \quad \phi \in \mathcal{H}.
\]

It is clear that \( T \in \mathcal{H}' \). Moreover, by the properties of the function \( K_{\mu} \) ([23])

\[
\begin{align*}
(T \#_{\mu} \phi)(x) &= \int_{-\infty}^{+\infty} \mu \tau_z(\phi)(-y) y^{2\mu} \, dy \\
&= \frac{1}{2^{\mu+1/2} \Gamma(\mu + 1/2)} \int_{-\infty}^{+\infty} \phi(z) \int_{-\infty}^{+\infty} K_{\mu}(x, -y, z) y^{2\mu} \, dy \, dz \\
&= \int_{-\infty}^{+\infty} \phi(z) z^{2\mu} \, dz, \quad \phi \in \mathcal{H}.
\end{align*}
\]

Hence, if \( \phi \in \mathcal{H} \) and \( \int_{-\infty}^{+\infty} \phi(z) z^{2m} \, dz \neq 0 \), the function \( e^{\mu x} T \#_{\mu} \phi \), \( m \in \mathbb{N} \setminus \{0\} \), is not bounded on \( \mathbb{R} \). Then, \( T \#_{\mu} \phi \) is not in \( \mathcal{H} \). However we prove that, for every \( T \in \mathcal{H}' \) and \( \phi \in \mathcal{H} \), \( T \#_{\mu} \phi \) is a multiplier of \( \mathcal{H} \).

**Proposition 3.2** Let \( T \in \mathcal{H}' \). Then, there exists \( l \in \mathbb{N} \) such that, for every \( k \in \mathbb{N} \) and \( \phi \in \mathcal{H} \),

\[
\sup_{x \in \mathbb{R}} e^{-|x|} |D^k (T \#_{\mu} \phi)(x)| < \infty.
\]

Hence, \( T \#_{\mu} \phi \in \mathcal{M}_\mathcal{H} \), for each \( \phi \in \mathcal{H} \).

**Proof.** According to [12, Proposition 3] there exist \( m \in \mathbb{N} \) and a bounded continuous function \( f \) on \( \mathbb{R} \) such that

\[
\langle T, \phi \rangle = \int_{-\infty}^{+\infty} e^{m|y|} f(y) D^m \phi(y) \, dy, \quad \phi \in \mathcal{H}.
\]
Let $\phi \in \mathcal{H}$. We can write, by (3.1),
\[
(T \#_\mu \phi)(x) = \int_{-\infty}^{+\infty} e^{i|y|} f(y) D^m_y (\mu \tau_x (\phi)(-y)) \, dy
= \int_{-\infty}^{+\infty} e^{i|y|} f(y) D^m_y (F_{\mu}^{-1} (e_{\mu}(ixz)F_{\mu}(\phi))(z)) \, dy, \quad x \in \mathbb{R}.
\]

Let $k \in \mathbb{N}$.

By using (2.6) we get that, for every $x \in \mathbb{R}$, $D^k_x e_{\mu}(ixz)$ is an entire function verifying
\[
|D^k_x e_{\mu}(ixz)| \leq C \mid z \mid^{|k|} e^{|z||\text{Im}z|}.
\]

Hence $D^k_x e_{\mu}(ixz)$ is a multiplier of $A$, for every $x \in \mathbb{R}$.

By invoking Theorem 2.3 there exist $l \in \mathbb{N}$ and $C > 0$ associated with $m$ such that
\[
sup_{y \in \mathbb{R}} e^{(m+1)|y|} \mid D^m_y (F_{\mu}^{-1} (D^k_x e_{\mu}(ixz)F_{\mu}(\phi))(z))(y) \mid
\leq C \sup_{|\text{Im}z| \leq l} (1 + |z|)^l \mid D^k_x e_{\mu}(ixz)\mid \mid F_{\mu}(\phi)(z) \mid
\leq C e^{l|z|} \sup_{|\text{Im}z| \leq l} (1 + |z|)^{l+k} \mid F_{\mu}(\phi)(z) \mid
\leq C e^{l|z|}, \quad x \in \mathbb{R}.
\]

Therefore we obtain
\[
|D^k_x (T \#_\mu \phi)(x)| \leq C e^{l|z|}, \quad x \in \mathbb{R},
\]
and the proof is finished.

Proposition 2.2 allows us to establish the following associative property for the distributional Dunkl convolution.

**Proposition 3.3** Let $T \in \mathcal{H}'$ and $\phi, \psi \in \mathcal{H}$. Then
\[
(T \#_\mu \phi) \#_\mu \psi = T \#_\mu (\phi \#_\mu \psi).
\]

**Proof.** According to Proposition 2.2 it is sufficient to prove the property when
\[
\langle T, \phi \rangle = \int_{-\infty}^{+\infty} e^{n|x|} D^k_x \phi(x) \, dw(x), \quad \phi \in \mathcal{H},
\]
where $n, k \in \mathbb{N}$ and $w$ is a complex measure on $\mathbb{R}$.

We can write
\[
T \#_\mu (\phi \#_\mu \psi)(x) = \int_{-\infty}^{+\infty} e^{n|t|} D^k_{\mu} (\mu \tau_x (\phi \#_\mu \psi)(-t)) \, dw(t), \quad x \in \mathbb{R}.
\]

By using (1.5) and (1.6) it follows that
\[
\mu \tau_x (\phi \#_\mu \psi) = (\mu \tau_x \phi) \#_\mu \psi, \quad x \in \mathbb{R},
\]
and
\[
\mu \tau_x (\phi)(y) = \mu \tau_y (\phi)(x), \quad x, y \in \mathbb{R}.
\]

Hence
\[
T \#_\mu (\phi \#_\mu \psi)(x) = \int_{-\infty}^{+\infty} e^{n|t|} D^k_{\mu} \int_{-\infty}^{+\infty} (\mu \tau_x \psi)(-z)(\mu \tau_{-t} \psi)(z) \frac{|z|^{2\mu}}{2^{\mu+1/2} \Gamma(\mu + 1/2)} \, dz \, dw(t) =
\]
From this property one deduces the following interchange formula for the Dunkl transform on $\mathcal{H}'$.

**Proposition 3.4** Let $T \in \mathcal{H}'$ and $\phi \in \mathcal{H}$. Then

\[ \mathcal{F}_\mu(T \#_\mu \phi) = \mathcal{F}_\mu(T \mathcal{F}_\mu \phi). \]

**Proof.** Let $\Phi \in \mathcal{A}$. By using Proposition 3.3, we have that

\[ \langle \mathcal{F}_\mu(T \#_\mu \phi), \Phi \rangle = \langle T \#_\mu \phi, \Phi^{-1} \mathcal{F}_\mu \rangle \]
\[ = \langle T, \#_\mu \Phi^{-1} \mathcal{F}_\mu \rangle \]
\[ = \langle \mathcal{F}_\mu(T), \mathcal{F}_\mu(\#_\mu \Phi^{-1} \mathcal{F}_\mu) \rangle \]
\[ = \langle \mathcal{F}_\mu(T), \mathcal{F}_\mu(\Phi) \rangle \]
\[ = \langle \mathcal{F}_\mu(T) \mathcal{F}_\mu(\Phi), \Phi \rangle. \]

Hence $\mathcal{F}_\mu(T \#_\mu \phi) = \mathcal{F}_\mu(T) \mathcal{F}_\mu(\phi)$. $\square$

Our next objective is to determine the elements of $\mathcal{H}'$ defining Dunkl convolution operators on $\mathcal{H}$.

Let $m \in \mathbb{Z}$, $m < 0$. We consider the space $O(\mathcal{H})_m$ that consists of all those smooth functions $\phi$ on $\mathbb{R}$ such that

\[ a^m_k(\phi) = \sup_{x \in \mathbb{R}} e^{m|x|} |D^k \phi(x)| < \infty, \]

for every $k \in \mathbb{N}$. $O(\mathcal{H})_m$ is endowed with the topology associated with the family $\{a^m_k\}_{k \in \mathbb{N}}$ of seminorms. Thus $O(\mathcal{H})_m$ is a Fréchet space.

For every $k \in \mathbb{N}$ and $\phi$ smooth on $\mathbb{R}$ we define $\beta^m_k(\phi)$ by

\[ \beta^m_k(\phi) = \sup_{x \in \mathbb{R}} e^{m|x|} |D^k \phi(x)|. \]

As in the proof of [14, Proposition 3.1] we can see that a smooth function $\phi$ on $\mathbb{R}$ is in $O(\mathcal{H})_m$ if and only if $\beta^m_k(\phi) < \infty$, and the family $\{\beta^m_k\}_{k \in \mathbb{N}}$ defines also the topology of $O(\mathcal{H})_m$. It is clear that $\mathcal{H}$ is contained in $O(\mathcal{H})_m$. We denote by $\theta(\mathcal{H})_m$ the closure of $\mathcal{H}$ in $O(\mathcal{H})_m$. By using standard arguments the following useful result can be established.

**Proposition 3.5** Let $\phi$ be a smooth function on $\mathbb{R}$ and $m \in \mathbb{Z}$, $m < 0$. Then the following assertions are equivalent.

(a) $\phi \in \theta(\mathcal{H})_m$.

(b) For every $k \in \mathbb{N}$, $\lim_{|x| \to \infty} e^{m|x|} |D^k \phi(x)| = 0$.

(c) For every $k \in \mathbb{N}$, $\lim_{|x| \to \infty} e^{m|x|} |D^k \phi(x)| = 0$.

The dual space of $\theta(\mathcal{H})_m$ is denoted as usual by $\theta(\mathcal{H})'_m$. As a consequence of Proposition 3.5 and by employing Hahn–Banach theorem and Riesz representation theorem we can obtain the following representation for the elements of $\theta(\mathcal{H})'_m$ (see, for instance, [4] for other similar results).
Proposition 3.6 Let $m \in \mathbb{Z}$, $m < 0$ and let $T$ be a functional on $\theta(\mathcal{H})_m$. Then, the following assertions are equivalent.

(a) $T \in \theta(\mathcal{H})'_m$.

(b) There exist $l \in \mathbb{N}$ and complex measures $\gamma_0, \ldots, \gamma_l$ on $\mathbb{R}$ such that
\[
\langle T, \phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} e^{m|x|} D^j_\mu \phi(x) d\gamma_j(x), \quad \phi \in \theta(\mathcal{H})_m.
\]

(c) There exist $l \in \mathbb{N}$ and complex measures $\gamma_0, \ldots, \gamma_l$ on $\mathbb{R}$ such that
\[
\langle T, \phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} e^{m|x|} D^j_\mu \phi(x) d\gamma_j(x), \quad \phi \in \theta(\mathcal{H})_m.
\]

If $m, l \in \mathbb{Z}$, $m < l < 0$, then $\theta(\mathcal{H})_l \subseteq \theta(\mathcal{H})_m$ and the inclusion is continuous.

We denote by $\theta(\mathcal{H})$ the union space $\bigcup_{m \in \mathbb{Z}, m < 0} \theta(\mathcal{H})_m$. The dual space of $\theta(\mathcal{H})$ is denoted by $\theta(\mathcal{H})'$ and we have that $\theta(\mathcal{H})' = \bigcap_{m \in \mathbb{Z}, m < 0} \theta(\mathcal{H})'_m$. According to Proposition 3.6 and [12, Proposition 9] we can identify $\theta(\mathcal{H})'$ with the space $\theta'_c(\Lambda_\infty, \Lambda_\infty)$ considered by Hasumi.

The next result characterizes $\theta(\mathcal{H})'$ as the space of Dunkl convolution operators on $\mathcal{H}$. It is remarkable that $\theta(\mathcal{H})'$ does not depend on $\mu$.

Theorem 3.7 Let $T \in \mathcal{H}'$. The following assertions are equivalent.

(a) $T \in \theta(\mathcal{H})'$.

(b) $F_\mu(T) \in \mathcal{M}_A$.

(c) For every $m \in \mathbb{N}$ there exist $l \in \mathbb{N}$ and $l + 1$ continuous functions $\{f_j\}_{j=0}^{l}$ such that
\[
\langle T, \phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} f_j(x) D^j_\mu \phi(x) |x|^{2m} dx, \quad \phi \in \mathcal{H},
\]

and $e^{m|x|} f_j, j = 0, \ldots, l,$ is a bounded function on $\mathbb{R}$.

(d) For every $\phi \in \mathcal{H}$, $T \#_\mu \phi \in \mathcal{H}$.

(e) The mapping $\phi \mapsto T \#_\mu \phi$ is continuous from $\mathcal{H}$ into itself.

Proof. (a) $\implies$ (b). Suppose that $T \in \theta(\mathcal{H})'$. Let $m \in \mathbb{N}$. According to Proposition 3.6, there exist $l \in \mathbb{N}$ and complex measures $\gamma_0, \ldots, \gamma_l$ on $\mathbb{R}$ for which
\[
\langle T, \phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} e^{-m|x|} D^j_\mu \phi(x) d\gamma_j(x), \quad \phi \in \mathcal{H}.
\]

Hence, for each $\Phi \in A$ we can write
\[
\langle F_\mu T, \Phi \rangle = \langle T, F_\mu^{-1} \Phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} e^{-m|x|} D^j_\mu (F_\mu^{-1}(\Phi)(x)) d\gamma_j(x) = \int_{-\infty}^{+\infty} \Phi(y) \sum_{j=0}^{l} (iy)^j \int_{-\infty}^{+\infty} e_{\mu}(ixy)(iy)^j \Phi(y) \frac{|y|^{2\mu}}{2^{2\mu+1/2} \Gamma(\mu + 1/2)} dy d\gamma_j(x) = \int_{-\infty}^{+\infty} \Phi(y) \sum_{j=0}^{l} (iy)^j \int_{-\infty}^{+\infty} e_{\mu}(ixy) d\gamma_j(x) \frac{|y|^{2\mu}}{2^{2\mu+1/2} \Gamma(\mu + 1/2)} dy.
\]
\[ F_\mu(T)(y) = \sum_{j=0}^{l} (i y)^j \int_{-\infty}^{+\infty} e^{-m|x|} e_{\mu}(i z x) \, d\gamma_j(x), \quad y \in \mathbb{R}. \] (3.3)

We observe that the right-hand side of (3.3) defines an analytic function on the strip \( \{ y \in \mathbb{C} : |\text{Im} \, y| < m \} \), for every \( m \in \mathbb{N} \). Then the arbitrariness of \( m \) allows us to conclude that \( F_\mu(T) \) is an entire function.

Moreover, if \( m \in \mathbb{N} \), by virtue of (2.8) we have that

\[
|F_\mu(T)(z)| \leq C \sum_{j=0}^{l} |z|^j \int_{-\infty}^{+\infty} e^{-m|x|} e|\text{Im} \, z| |x| |d\gamma_j(x)| \leq C(1 + |z|)^l \sum_{j=0}^{l} |\gamma_j|(\mathbb{R}), \quad |\text{Im} \, z| \leq m,
\]

for certain \( l \in \mathbb{N} \) and complex measures \( \gamma_0, \ldots, \gamma_l \). Hence, \( F_\mu(T) \in \mathcal{M}_A \) and (b) is established.

(b) \( \Rightarrow \) (c). Assume now that \( F = F_\mu(T) \) is a multiplier of \( A \). Then \( F \) is an entire function and, for every \( k \in \mathbb{N} \) there exists \( m_k \in \mathbb{N} \) for which

\[
\sup_{|\text{Im} \, z| \leq k} (1 + |z|)^{-m_k} |F(z)| < \infty. \tag{3.4}
\]

Let \( m, k \in \mathbb{N} \), \( k > m \) and let \( m_k \in \mathbb{N} \) be the value given in (3.4). We define the function

\[ G_l(z) = \frac{F(z)}{(z^2 + (k + 2)^2)^l}, \quad |\text{Im} \, z| < k + 2, \]

where \( l \in \mathbb{N} \) will be specified later.

Note that \( G_l \) is a holomorphic function on the strip \( \{ z \in \mathbb{C} : |\text{Im} \, z| < k + 2 \} \). Moreover, if we choose \( l \in \mathbb{N} \) such that \( 2l > m_k + 2\mu + 1 \) then \( G_l \in L^1(\mathbb{R}, |x|^{2\mu} \, dx) \). Indeed, according to (3.4) we can write

\[
\int_{-\infty}^{+\infty} |G_l(x)||x|^{2\mu} \, dx \leq C \int_{-\infty}^{+\infty} \frac{|x|^{2\mu}}{(1 + |x|)^{2l - m_k}} \, dx < \infty.
\]

Hence \( \mathcal{F}_\mu^{-1}(G_l) \) as element of \( \mathcal{H}' \) coincides, according to (2.4), with the function \( \mathcal{F}_\mu(G_l) \) as element of \( \mathcal{H}' \) in the sense of (2.4). In fact, for every \( \phi \in \mathcal{H} \), we have

\[
\langle \mathcal{F}_\mu^{-1}(G_l), \phi \rangle = \langle G_l, \mathcal{F}_\mu \phi \rangle = \int_{-\infty}^{+\infty} G_l(x) \mathcal{F}_\mu(\phi)(x) \frac{|x|^{2\mu}}{2^{\mu + 1/2} \Gamma(\mu + 1/2)} \, dx = \int_{-\infty}^{+\infty} \phi(y) \mathcal{F}_\mu(G_l)(y) \frac{|y|^{2\mu}}{2^{\mu + 1/2} \Gamma(\mu + 1/2)} \, dy.
\]

Thus, if \( 2l > m_k + 2\mu + 1 \), we obtain

\[
T = \mathcal{F}_\mu^{-1} \left( (x^2 + (k + 2)^2)^l G_l(x) \right) = \sum_{j=0}^{l} \epsilon_{j,k}^{1,l} \mathcal{F}_\mu^{-1}(x^2) G_l(x) = \sum_{j=0}^{l} (-1)^{j} \epsilon_{j,k}^{1,l} D_{2\mu}^j (\mathcal{F}_\mu^{-1}(G_l)),
\]
for certain constants $c_{j}^{(l)}$, $j = 0, \ldots, l$. Here all the equalities must be understood of course in a distributional sense. Then we can write

$$ T = \sum_{j=0}^{l} D_{\mu}^{2j} f_{j}, $$

where $f_{j} = (-1)^{j} c_{j}^{(l)} F_{\mu}(G_{l})$, is a function for $j = 0, \ldots, l$. Here $D_{\mu}$ is understood again in a distributional sense.

Then, to establish (c) we must only prove that $e^{m|x|} F_{\mu}^{-1}(G_{l})$ is a bounded function on $\mathbb{R}$ for certain $l \in \mathbb{N}$. To this aim we proceed by using the technique employed by Anker [1].

Let $g_{l} = F_{\mu}^{-1}(G_{l})$. According to (2.1) we get

$$ \sup_{|x| \leq 1} e^{m|x|} |g_{l}(x)| \leq C \int_{-\infty}^{+\infty} |G_{l}(y)||y|^{2\mu} dy < \infty, $$

(3.5)

provided that $2l > m_{k} + 2\mu + 1$.

Let $h_{l} = F_{0}^{-1}(G_{l})$. This function satisfies the following property. For every $s, l \in \mathbb{N}$ such that $2l > s + m_{k} + 1$ we have that

$$ \left| \frac{d^s}{dt^s} h_{l}(t) \right| \leq C e^{-(m+2)|t|}, \quad t \in \mathbb{R}. $$

(3.6)

In fact, let $s, l \in \mathbb{N}$ be such that $2l > s + m_{k} + 1$. By using Cauchy’s Theorem to change the path of integration we get

$$ \frac{d^s}{dt^s} h_{l}(t) = \frac{i^{s}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\beta y} \frac{y^{s} F(y)}{(y^{2} + (k + 2)^{2})} dy $$

$$ = \frac{i^{s}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\beta y} \frac{y^{s} F(y)}{(y^{2} + (k + 2)^{2})} dy, \quad t \in \mathbb{R}, $$

where $\beta = (m + 2)\text{sgn}(t)$, $t \in \mathbb{R}$, where $\text{sgn}(t)$ is the sign function.

Hence, according to (3.4)

$$ \left| \frac{d^s}{dt^s} h_{l}(t) \right| \leq C e^{-\beta \gamma} \int_{-\infty}^{+\infty} (1 + |x|)^{s + m_{k} - 2l} dx \leq C e^{-(m + 2)|t|}, \quad t \in \mathbb{R}, $$

and (3.6) is established. Note that by virtue of (3.6) $h_{l} \in L^{1}(\mathbb{R}, dx)$ and hence $G_{l} = F_{0}(h_{l})$, whenever $2l > m_{k} + 1$.

As in the proof of Theorem 2.3, for each $n \in \mathbb{N}$, $n \geq 1$, we choose a smooth function $w_{n}$ such that $w_{n}(x) = 1$, $|x| \leq n$, $w_{n}(x) = 0$, $|x| \geq n + 1$ and such that, for every $r \in \mathbb{N}$ there exists $C_{r} > 0$ (that does not depend on $n$) for which

$$ |D^{r} w_{n}(x)| \leq C_{r}, \quad x \in \mathbb{R}. $$

(3.7)

We consider the decomposition $h_{l} = w_{n} h_{l} + (1 - w_{n}) h_{l}$ and define $h_{l,n} = (1 - w_{n}) h_{l}$, $G_{l,n} = F_{0}(h_{l,n})$ and $g_{l,n} = F_{\mu}^{-1}(G_{l,n})$, $n \in \mathbb{N}$, $n \geq 1$. Note that, by (3.6) and (3.7), $G_{l,n} \in L^{1}(\mathbb{R}, |x|^{2\mu} dx)$, provided that $2l > 2\mu + 3 + m_{k}$. Indeed, it is sufficient to see that if $2\gamma = [2\mu] + 2$,

$$ \sup_{x \in \mathbb{R}} (1 + x^{2})^{\gamma} |G_{l,n}(x)| \leq C \sum_{i=0}^{2l} \int_{-\infty}^{+\infty} \left| \frac{d^i}{dy^i} h_{l,n}(y) \right| dy < \infty. $$

We have that $h_{l}(t) = h_{l,n}(t)$, $|t| > n + 1$. Thus according to (3.6) if $s, l \in \mathbb{N}$, $2l > s + m_{k} + 1$ then

$$ \frac{d^s}{dt^s} h_{l,n}(t) \to 0, \quad \text{as} \quad |t| \to \infty. $$

(3.8)
On the other hand, since \( \text{supp}(w_n h_l) \subseteq [-n - 1, n + 1] \), by virtue of Paley–Wiener theorems for Euclidean Fourier transform \( \mathcal{F}_0 \) and Dunkl transforms (see [1, Proposition 3 and Theorem 10]) we have that \( \text{supp}(g_l - g_{l,n}) \subseteq [-n - 1, n + 1] \), that is, \( g_l(x) = g_{l,n}(x), |x| > n + 1 \).

Let \( n \in \mathbb{N}, n \geq 2 \). We obtain

\[
\sup_{n+1 \leq |x| \leq n+2} e^{m|x|} |g_l(x)| = \sup_{n+1 \leq |x| \leq n+2} e^{m|x|} |g_{l,n}(x)| \leq C e^{m} \sup_{n+1 \leq |x| \leq n+2} \int_{-\infty}^{+\infty} |e_\mu(ixy)||G_{l,n}(y)||y|^{2n} \, dy 
\]

\[
\leq C e^{m} \sup_{y \in \mathbb{R}} (1 + y^2) |G_{l,n}(y)|, 
\]

where \( r > \mu + 1/2 \).

Let us consider \( r > \mu + 1/2 \) and \( l \in \mathbb{N} \) such that \( 2l > 2r + m_k + 1 \). We can write

\[
(1 + y^2)^r G_{l,n}(y) = \sum_{j=0}^{r} \binom{r}{j} y^{2j} \mathcal{F}_0(h_{l,n})(y) 
\]

\[
= \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \int_{-\infty}^{+\infty} e^{-jt} \frac{d^{2j}}{dt^{2j}} h_{l,n}(t) \, dt, \quad y \in \mathbb{R}. 
\]

By integration by parts and taking into account (3.8) one gets

\[
(1 + y^2)^r G_{l,n}(y) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \int_{-\infty}^{+\infty} e^{-jt} \frac{d^{2j}}{dt^{2j}} h_{l,n}(t) \, dt, \quad y \in \mathbb{R}. 
\]

Thus, according to (3.7) and since \( h_{l,n}(x) = 0, |x| \leq n \), we get

\[
\sup_{y \in \mathbb{R}} |(1 + y^2)^r G_{l,n}(y)| \leq C \sum_{j=0}^{r} \int_{-\infty}^{+\infty} \left| \frac{d^{2j}}{dt^{2j}} h_{l,n}(t) \right| \, dt
\]

\[
\leq C \sum_{j=0}^{r} \left( 1 + t^2 \right) \left| \frac{d^{2j}}{dt^{2j}} h_{l,n}(t) \right| 
\]

\[
\leq C \sum_{j=0}^{r} \sum_{s=0}^{2j} \sup_{|t| \geq n} \left( 1 + t^2 \right) \left| \frac{d^s}{dt^s} h_l(t) \right| 
\]

\[
\leq C \sum_{j=0}^{r} \sum_{s=0}^{2j} \sup_{|t| \geq n} e^{2s} \left| \frac{d^s}{dt^s} h_l(t) \right|. 
\]

Then property (3.6) allows us to conclude that

\[
\sup_{y \in \mathbb{R}} |(1 + y^2)^r G_{l,n}(y)| \leq C \sup_{|t| \geq n} e^{-m|t|},
\]

and hence,

\[
\sup_{n+1 \leq |x| \leq n+2} e^{m|x|} |g_l(x)| \leq C, \tag{3.9}
\]

where \( C \) is a constant not depending on \( n \). By (3.5) and (3.9) we conclude that \( e^{m|x|} g_l \) is a bounded function when we choose \( l \in \mathbb{N} \) such that \( 2l > 2\mu + m_k + 4 \).

(e) \implies (a). Suppose that (e) is verified. Let \( m \in \mathbb{N}, m \geq 1 \). We can write

\[
\langle T, \phi \rangle = \sum_{j=0}^{l} \int_{-\infty}^{+\infty} f_j(x) D_\mu^j \phi(x) |x|^{2n} \, dx, \quad \phi \in \mathcal{H}, \tag{3.10}
\]
for certain \(l \in \mathbb{N}\) and \(\{f_j\}_{j=0}^l\) continuous functions such that \(e^{m|x|}f_j, j = 0, \ldots, l\), is a bounded function on \(\mathbb{R}\). Then,

\[
|\langle T, \phi \rangle| \leq \sum_{j=0}^l \int_{-\infty}^{+\infty} e^{-m|x|} |D_j^\mu \phi(x)||x|^{2n} \, dx \leq C \sum_{j=0}^l \beta_j^{-(m-1)}(\phi), \quad \phi \in \mathcal{H}.
\]

Hence we prove that \(T\) can be extended to \(\theta(\mathcal{H})_{-m}\) as an element of \(\theta(\mathcal{H})_{-m}\) in a unique way by the right-hand side of (3.10).

\((e) \implies (d)\). It is clear.

\((d) \implies (e)\). Assume \((d)\) is satisfied. We will use the Closed Graph Theorem to establish \((e)\).

Let \((\phi_n)_{n=1}^\infty \subset \mathcal{H}\) and let \(\phi, \psi \in \mathcal{H}\) be such that \(\phi_n \to \phi\) and \(T\#_\mu \phi_n \to \psi\), as \(n \to \infty\), in the topology of the space \(\mathcal{H}\).

Since \(\phi \to \tau_x \phi, x \in \mathbb{R}\), is a continuous mapping from \(\mathcal{H}\) into itself and \(T \in \mathcal{H}'\) we have that \(T\#_\mu \phi_n(x) \to T\#_\mu \phi(x), x \in \mathbb{R}\).

On the other hand, since \(T\#_\mu \phi_n \to \psi\), as \(n \to \infty\), in \(\mathcal{H}\), we deduce that \(T\#_\mu \phi_n \to \psi\) uniformly in \(\mathbb{R}\).

Thus, \(T\#_\mu \phi(x) = \psi(x), x \in \mathbb{R}\), and the Closed Graph Theorem allows us to conclude \((e)\).

\((b) \implies (e)\). Assume \(\mathcal{F}_\mu(T) \in \mathcal{MA}\). By the interchange formula (3.2) and Theorem 2.3 we have that \(\mathcal{F}_\mu(T\#_\mu \phi) \in \mathcal{A}\) and \(T\#_\mu \phi = \mathcal{F}_\mu^{-1}(\mathcal{F}_\mu T\mathcal{F}_\mu \phi), \phi \in \mathcal{H}\).

Hence, Theorem 2.3 leads to \((e)\).

\((e) \implies (b)\). Theorem 2.3 and property \((e)\) imply that \(\mathcal{F}_\mu(T) \in \mathcal{A}'\) and the mapping \(\Phi \to \Phi \mathcal{F}_\mu(T)\) is continuous from \(\mathcal{A}\) into itself. Hence \(\mathcal{F}_\mu(T) \in \mathcal{MA}\).

Let \(T \in \mathcal{H}'\) and \(S \in \theta(\mathcal{H})'\). Theorem 3.7 allows us to define the Dunkl convolution \(T\#_\mu S\) of \(T\) and \(S\) as the element of \(\mathcal{H}'\) given by

\[
\langle T\#_\mu S, \phi \rangle = \langle T, S\#_\mu \phi \rangle = \langle T(x), \langle S(y), \mu \tau_x \phi(-y) \rangle \rangle, \quad \phi \in \mathcal{H}.
\]

The following interchange formula for this distributional convolution is obtained easily from Proposition 3.4.

**Proposition 3.8** Let \(T \in \mathcal{H}'\) and \(S \in \theta(\mathcal{H})'\). Then

\[
\mathcal{F}_\mu(T\#_\mu S) = \mathcal{F}_\mu(T) \mathcal{F}_\mu(S).
\]

Also, the following algebraic properties for the distributional Dunkl convolution are satisfied.

**Proposition 3.9** Let \(T \in \mathcal{H}'\) and \(R, S \in \theta(\mathcal{H})'\). Then

(a) \(R\#_\mu S \in \theta(\mathcal{H})'\) and \(R\#_\mu S = S\#_\mu R\).

(b) \(T\#_\mu (R\#_\mu S) = (T\#_\mu R)\#_\mu S\).

(c) \(D_\mu(T\#_\mu R) = (D_\mu T)\#_\mu R = T\#_\mu D_\mu R\). Here \(D_\mu\) is understood in a distributional sense.

(d) \(T\#_\mu \delta = T\), where \(\delta\) is the Dirac functional.

**Proof.** Properties \((b)\) and \((c)\) follow immediately from Proposition 3.8 and to prove \((a)\) and \((d)\) it is sufficient to consider also Theorem 3.7 and that \(\mathcal{F}_\mu \delta = 1\).

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