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Oblique wave scattering by a circular cylinder submerged beneath an ice-cover

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Abstract

When a train of small-amplitude surface water waves is normally incident on a very long horizontal circular cylinder fully submerged in deep water with a *free surface*, it is well known that it passes over and below the cylinder with a change of phase without experiencing any reflection. However the cylinder does experience reflection for oblique incidence of the surface wave train. It is shown here that the same phenomenon also holds good when the deep water has an *ice-cover* instead of a free surface, the ice-cover being modelled as a thin elastic plate. Here, for oblique incidence, the reflection and transmission coefficients are obtained approximately and depicted graphically against the wave number in a number of figures.

Keywords: Wave scattering; Ice-covered water; Reflection and transmission coefficients

1. Introduction

There is a classical result in the linearised theory of water waves which states that a long circular cylinder with horizontal axis completely submerged in deep water with a free surface, experiences no reflection (cf. Dean [1], Ursell [2], Ogilive [3]) when a train of surface water waves is normally incident on it. However, for oblique incidence of the wave train, the cylinder does experience reflection (cf. Levine [4]). In recent times there is a considerable interest in the study of various types of water wave problems in the presence of a thin ice-sheet floating on water, the ice-sheet being modelled as a thin elastic plate (cf. Fox and Squire [5], Chakrabarti [6], Chung and Fox [7], Linton and Chung [8], Mandal and Basu [9], Gayen et al. [10], Mandal and Maiti [11]).This has motivated us to investigate on here the problem of scattering of obliquely incident waves by a long circular cylinder submerged beneath a thin ice-sheet floating on deep water, the ice-sheet being modelled as an elastic plate as mentioned above. By an appropriate use of Green's integral theorem applied to the scattered potential and a suitably constructed Green's function, a representation of the scattered potential

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function at a point in the fluid region is obtained in terms of an integral involving the unknown scattered potential on the surface of the cylinder. This unknown function satisfies an integral equation of second kind on the circular boundary of the cylinder. When this potential function is replaced by its equivalent general Fourier series in the angular co-ordinate with origin at the center of the circular cross-section, two linear infinite systems involving the two sets of Fourier coefficients are obtained after using the orthogonal properties of trigonometric functions. For the case of normal incidence, the two linear systems become identical, as a consequence of which, the reflection coefficient is seen to vanish identically for all cylinder sizes (i.e., radius of the cylinder), depth of its submergence and incident wave frequencies. For oblique incidence, the two linear systems are however different, and as such the reflection coefficient does not vanish. The two infinite linear systems can be shown to possess unique solutions, and hence can be solved approximately by truncation. Both the reflection and transmission coefficients are obtained approximately up to first and second-order in terms of computable integrals. Numerical estimates for these quantities are obtained and are depicted graphically against the wave number for various values of the angle of incidence and other parameters in a number of figures. For small angle of incidence, the reflection coefficient is seen to be negligible, which is expected.

2. Mathematical formulation

Let a circular cylinder represented by $r = (x^2 + y^2)^{\frac{1}{2}} = a$ be submerged fully in an infinitely deep water occupying the region $y \geqslant -h$, $-\infty < x < \infty$, $-\infty < z < \infty$, $r \geqslant a$, where $h(>a)$ is the depth of submergence of the center of a cross-section of the cylinder, a its is radius, and y-axis is taken vertically downwards. On the surface of water floats a very thin ice-sheet which is modelled as an elastic plate. Let a harmonically time-dependent train of waves described by the potential function $Re\{\phi_0(x,y)e^{-i\sigma t+i\gamma z}\}$ be obliquely incident on the fixed circular cylinder. Here $\gamma = \lambda \sin\alpha$, where α is the angle of incidence of the wave train and σ is the angular frequency, and λ is the unique positive real root of the dispersion equation

$$k(Dk^4 + 1 - \epsilon K) = K. \tag{2.1}$$

In (2.1) $K = \frac{\sigma^2}{g}$, g being the acceleration due to gravity, D is the flexural rigidity of the ice-cover and $\epsilon = \frac{\rho_0}{\rho_1} h_0$, ρ_0 and ρ_1 are the densities of ice and water, respectively and h_0 is the very small thickness of the ice-cover. The other roots of (2.1) are $\lambda_1, \overline{\lambda_1}$ and $\lambda_2, \overline{\lambda_2}$ where $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, and

$$\phi_0(x,y) = e^{-\lambda y + i\mu x}, \tag{2.2}$$

with $\mu = \lambda\cos\alpha$.

The velocity potential function describing the resulting motion in the water can be represented by $Re\{\phi(x,y)e^{-i\sigma t+i\gamma z}\}$, where the complex valued potential function $\phi(x,y)$ satisfies

$$(\nabla^2 - \gamma^2)\phi = 0, \quad r > a, \quad y > -h, \quad -\infty < x < \infty, \tag{2.3}$$

$$\left\{D\left(\frac{\partial^2}{\partial x^2} - \gamma^2\right)^2 + 1 - \epsilon K\right\}\phi_y + K\phi = 0 \quad \text{on } y = -h, \tag{2.4}$$

$$\frac{\partial\phi}{\partial r} = 0 \quad \text{on } r = a, \tag{2.5}$$

where $x = r\sin\theta$, $y = r\cos\theta$ $(-\pi \leqslant \theta \leqslant \pi)$,

$$\nabla\phi \to 0 \quad \text{as } y \to \infty, \tag{2.6}$$

and

$$\phi(x,y) \sim \begin{cases} T\phi_0(x,y) & \text{as } x \to \infty, \\ \phi_0(x,y) + R\phi_0(-x,y) & \text{as } x \to -\infty. \end{cases} \tag{2.7}$$

In (2.7) T and R denote respectively the transmission and reflection coefficients and are unknown complex constants to be determined.

3. Solution of the problem

Let $G(x,y;\xi,\eta)$ denote the Green's function satisfying (2.3) except at (ξ,η) $(\eta > -h)$ with boundary conditions (2.4), (2.6) and represent an outgoing wave at infinity. We can represent $G(x,y;\xi,\eta)$ as

$$G(x,y;\xi,\eta) = K_0(\gamma r_1) - K_0(\gamma r_2) + \int_\gamma^\infty A(k)\mathrm{e}^{-k(y+h)} \cos\left\{\left(k^2-\gamma^2\right)^{\frac{1}{2}}(x-\xi)\right\}\mathrm{d}k, \tag{3.1}$$

where $K_0(x)$ denotes the modified Bessel function of second kind, and

$$r_1 = ((x-\xi)^2 + (y-\eta)^2)^{\frac{1}{2}}, \quad r_2 = ((x-\xi)^2 + (y+\eta+2h)^2)^{\frac{1}{2}}$$

and $A(k)$ is a function of k to be found such that the integral exits in some sense. The ice-cover condition (2.4) is satisfied if $A(k)$ is chosen as

$$A(k) = 2\frac{k\left(Dk^4+1-\epsilon K\right)}{\left(k^2-\gamma^2\right)^{\frac{1}{2}}\left\{k(Dk^4+1-\epsilon K)-K\right\}}\mathrm{e}^{-k(\eta+h)}. \tag{3.2}$$

Thus we can take

$$G(x,y;\xi,\eta) = K_0(\gamma r_1) - K_0(\gamma r_2) + 2⨍_\gamma^\infty \frac{k\left(Dk^4+1-\epsilon K\right)}{\left(k^2-\gamma^2\right)^{\frac{1}{2}}\left\{k(Dk^4+1-\epsilon K)-K\right\}}\mathrm{e}^{-k(y+\eta+2h)} \times \cos\left\{\left(k^2-\gamma^2\right)^{\frac{1}{2}}(x-\xi)\right\}\mathrm{d}k \tag{3.3}$$

where the contour is indented below the pole $k = \lambda$ on the real k-axis to take care of its outgoing nature at infinity. An alternative representation in which its behavior as $(x-\xi) \to \pm\infty$ is evident is given by

$$\begin{aligned} G(x,y;\xi,\eta) = {} & K_0(\gamma r_1) - K_0(\gamma r_2) + 2\pi\mathrm{i}\left[\frac{\lambda g(\lambda)}{\cos\alpha}\mathrm{e}^{-\lambda(y+\eta+2h)+\mathrm{i}\left(\lambda^2-\gamma^2\right)^{\frac{1}{2}}|x-\xi|} + \frac{\lambda_1 g(\lambda_1)}{\left(\lambda_1^2-\gamma^2\right)^{\frac{1}{2}}}\mathrm{e}^{-\lambda_1(y+\eta+2h)+\mathrm{i}\left(\lambda_1^2-\gamma^2\right)^{\frac{1}{2}}|x-\xi|}\right. \\ & \left. + \frac{\overline{\lambda_1}g(\overline{\lambda_1})}{\left(\overline{\lambda_1}^2-\gamma^2\right)^{\frac{1}{2}}}\mathrm{e}^{-\overline{\lambda_1}(y+\eta+2h)+\mathrm{i}\left(\overline{\lambda_1}^2-\gamma^2\right)^{\frac{1}{2}}|x-\xi|}\right] \\ & + 2\int_0^\infty \frac{k(Dk^4+1-\epsilon K)\left[k(Dk^4+1-\epsilon K)\cos k(y+\eta+2h) - K\sin k(y+\eta+2h)\right]}{\left(k^2+\gamma^2\right)^{\frac{1}{2}}\left\{k^2(Dk^4+1-\epsilon K)^2+K^2\right\}} \\ & \times \mathrm{e}^{-\left(k^2+\gamma^2\right)^{\frac{1}{2}}|x-\xi|}\mathrm{d}k \end{aligned} \tag{3.4}$$

where

$$g(\lambda) = \frac{D\lambda^4+1-\epsilon K}{5D\lambda^4+1-\epsilon K}. \tag{3.5}$$

To obtain a representation of $\phi(x,y)$ at a point (ξ,η), we apply Green's integral theorem to the functions $\psi(x,y) = \phi(x,y) - \phi_0(x,y)$ and the Green's function $G(x,y;\xi,\eta)$ within the entire fluid domain surrounding the rigid cylinder. This gives

$$2\pi\psi(\xi,\eta) = -\int_{-\pi}^{\pi}\psi(\theta)\left\langle a\frac{\partial}{\partial r}G(x,y;\xi,\eta)\right\rangle\mathrm{d}\theta - \int_{-\pi}^{\pi}G(x,y;\xi,\eta)\left\langle a\frac{\partial}{\partial r}\phi_0(x,y)\right\rangle\mathrm{d}\theta, \tag{3.6}$$

where $\psi(\theta)$ is the unknown scattered potential function on the contour of the cylinder and the angular bracket denotes the values at $r = a$.

By another use of Green's integral theorem to $\psi(x,y)$ and $G(x,y;a\sin\beta,a\cos\beta)$ in the fluid region with a small indentation at the point $(a\sin\beta,a\cos\beta)$ on the circle $r = a$, $\psi(\theta)$ can be shown to satisfy an integral equation of the second kind given by

$$\pi\psi(\beta) = -\int_{-\pi}^{\pi}\psi(\theta)\left\langle a\frac{\partial G}{\partial r}\right\rangle \mathrm{d}\theta - \int_{-\pi}^{\pi} G(x,y;a\sin\beta,a\cos\beta)\left\langle a\frac{\partial}{\partial r}\phi_0(x,y)\right\rangle \mathrm{d}\theta. \tag{3.7}$$

Making $\xi \to \mp\infty$ in (3.6), we get

$$R, T-1 = -\mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[\int_{-\pi}^{\pi}\psi(\theta)\left\langle a\frac{\partial}{\partial r}\mathrm{e}^{-\lambda y\pm \mathrm{i}\mu x}\right\rangle \mathrm{d}\theta + \int_{-\pi}^{\pi}\mathrm{e}^{-\lambda y\pm\mathrm{i}\mu x}\left\langle a\frac{\partial}{\partial r}\phi_0(x,y)\right\rangle \mathrm{d}\theta\right]. \tag{3.8}$$

Now we expand $\psi(\theta)$ as

$$\psi(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta), \quad -\pi<\theta<\pi, \tag{3.9}$$

where a_0, a_n, b_n $(n=1,2,\ldots)$ are unknown Fourier coefficients. Using (3.9) and multiplying both sides of (3.7) by $\cos s\beta, \sin s\beta$, respectively and integrating with respect to β from $-\pi$ to π, the following two infinite linear systems for a_n, b_n are obtained:

$$\pi^2 a_s + \sum_{n=0}^{\infty}a_nP_{ns}^{(1)} + \sum_{n=1}^{\infty}b_nP_{ns}^{(2)} + \sum_{n=0}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}B_{ns}^{(1)} = V_s^{(1)}, \quad s=0,1,2,\ldots \tag{3.10}$$

$$\pi^2 b_s + \sum_{n=0}^{\infty}a_nP_{ns}^{(3)} + \sum_{n=1}^{\infty}b_nP_{ns}^{(4)} + \sum_{n=0}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}B_{ns}^{(2)} = V_s^{(2)}, \quad s=1,2,\ldots \tag{3.11}$$

where

$$P_{ns}^{(1),(2)} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}G(x,y;a\sin\beta,a\cos\beta)\right\rangle \begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}\cos s\beta\,\mathrm{d}\theta\,\mathrm{d}\beta, \tag{3.12}$$

$$P_{ns}^{(3),(4)} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}G(x,y;a\sin\beta,a\cos\beta)\right\rangle \begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}\sin s\beta\,\mathrm{d}\theta\,\mathrm{d}\beta, \tag{3.13}$$

$$B_{ns}^{(1),(2)} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}G(x,y;a\sin\beta,a\cos\beta)\right\rangle(\cos\theta - \mathrm{i}\cos\alpha\sin\theta)^n \begin{matrix}\cos s\beta\\ \sin s\beta\end{matrix}\mathrm{d}\theta\,\mathrm{d}\beta, \tag{3.14}$$

$$V_s^{(1),(2)} = -\pi\int_{-\pi}^{\pi}\phi_0(\xi,\eta)\begin{matrix}\cos s\beta\\ \sin s\beta\end{matrix}\mathrm{d}\beta. \tag{3.15}$$

Since

$$\int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}G(x,y;\xi,\eta)\right\rangle \mathrm{d}\theta = 0,$$

it is obvious that there is no effect of a_0 on the function $\psi(\xi,\eta)$ in (3.6).

Then R and T reduce to

$$R, T-1 = -\mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[\sum_{n=1}^{\infty}a_n(E_n \pm \mathrm{i}p_n) + \sum_{n=1}^{\infty}b_n(q_n \pm \mathrm{i}H_n) + \sum_{n=0}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}\int_{-\pi}^{\pi}\left\langle a\frac{\partial\phi_0}{\partial r}\right\rangle(\cos\theta \mp \mathrm{i}\cos\alpha\sin\theta)^n\mathrm{d}\theta\right], \tag{3.16}$$

where

$$E_n, p_n = \int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}\begin{matrix}\cos\mu x\\ \sin\mu x\end{matrix}\mathrm{e}^{-\lambda y}\right\rangle\cos n\theta\,\mathrm{d}\theta,$$

$$q_n, H_n = \int_{-\pi}^{\pi}\left\langle a\frac{\partial}{\partial r}\begin{matrix}\cos\mu x\\ \sin\mu x\end{matrix}\mathrm{e}^{-\lambda y}\right\rangle\sin n\theta\,\mathrm{d}\theta.$$

To evaluate these integrals, we use the following result given by Levine [4]:

Let $u(x, y)$ satisfy

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\lambda^2\sin^2\alpha\right)u(x,y)=0,\quad \text{for } 0\leqslant r\leqslant a \tag{3.17}$$

and be regular for $0 \leqslant r \leqslant a$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\partial}{\partial r}u(r,\theta)\right]_{r=a}\begin{matrix}\cos n\theta\\ \sin n\theta\end{matrix}\,\mathrm{d}\theta=(-1)^n\left[\frac{\mathrm{d}}{\mathrm{d}r}I_n(\lambda r\sin\alpha)\right]_{r=a}\begin{matrix}\cos nD\\ \sin nD\end{matrix}\,u(0), \tag{3.18}$$

where

$$\cos D=-\frac{1}{\lambda\sin\alpha}\frac{\partial}{\partial y},\quad \sin D=-\frac{1}{\lambda\sin\alpha}\frac{\partial}{\partial x}$$

and $x = r\sin\theta, y = r\cos\theta$, $-\pi \leqslant \theta \leqslant \pi$, $I_n(z)$ being the modified Bessel function of first kind.

Using this result, it can be shown that

$$p_n=q_n=0, \tag{3.19}$$
$$P_{ns}^{(2)}=P_{ns}^{(3)}=0. \tag{3.20}$$

Thus the Eqs. (3.10) and (3.11) reduce to

$$\pi^2 a_s+\sum_{n=1}^{\infty}a_nP_{ns}^{(1)}+\sum_{n=1}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}B_{ns}^{(1)}=V_s^{(1)},\quad s=1,2,\ldots \tag{3.21}$$

$$\pi^2 b_s+\sum_{n=1}^{\infty}b_nP_{ns}^{(4)}+\sum_{n=1}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}B_{ns}^{(2)}=V_s^{(2)},\quad s=1,2,\ldots \tag{3.22}$$

as the constants $B_{01}^{(1)}$, $B_{01}^{(2)}$ can be shown to be zero. In the Appendix A, it is shown that the infinite linear systems (3.21) and (3.22) possess unique solutions.

Thus R and T in (3.16) can be written in the forms

$$R,T-1=-\mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[\sum_{n=1}^{\infty}(a_nE_n\pm \mathrm{i}b_nH_n)+\sum_{n=0}^{\infty}(-1)^n\frac{(\lambda a)^n}{n!}\int_{-\pi}^{\pi}\left\langle a\frac{\partial\phi_0}{\partial r}\right\rangle(\cos\theta\mp \mathrm{i}\cos\alpha\sin\theta)^n\mathrm{d}\theta\right]. \tag{3.23}$$

Now for the case of normal incidence, $\alpha = 0$, and in this case, it can be shown that

$$E_n=-H_n,\quad n=0,1,2,\ldots,$$
$$P_{ns}^{(1)}=P_{ns}^{(4)},\quad n,s=1,2,\ldots,$$

and

$$V_s^{(1)}=\mathrm{i}V_s^{(2)},\quad s=1,2,\ldots.$$

Thus for $\alpha = 0$, the Eqs. (3.21) and (3.22) reduce to

$$\pi^2A_s^{(1)}+\sum_{n=1}^{\infty}P_{ns}^{(1)}A_n^{(1)}=2(-1)^s\frac{\pi^2(a\lambda)^s}{s!},\quad s=1,2,\ldots \tag{3.24}$$

$$\pi^2A_s^{(2)}+\sum_{n=1}^{\infty}P_{ns}^{(1)}A_n^{(2)}=2(-1)^s\frac{\pi^2(a\lambda)^s}{s!},\quad s=1,2,\ldots \tag{3.25}$$

where

$$\begin{pmatrix} A_n^{(1)} \\ A_n^{(2)} \end{pmatrix} = \begin{pmatrix} a_n \\ \mathrm{i}b_n \end{pmatrix} + (-1)^n \frac{(a\lambda)^n}{n!}. \tag{3.26}$$

It is noted that the two linear systems (3.24) and (3.25) are exactly the same, and hence $A_n^{(1)} = A_n^{(2)}$ so that

$$a_n = \mathrm{i}b_n, \quad n = 1, 2, \ldots \tag{3.27}$$

Thus R and $T - 1$ reduce to

$$R, T - 1 = -\mathrm{i}g(\lambda)\mathrm{e}^{-2\lambda h} \sum_{n=1}^{\infty} \left(A_n^{(1)} E_n \pm A_n^{(2)} H_n\right)$$

and this produces for normal incidence of the wave train, the classical result

$$R \equiv 0.$$

This result holds for all frequencies and radius of the submerged cylinder. Hence, as in the case of a horizontal circular cylinder submerged beneath a free surface, here also the circular cylinder experiences no reflection for normally incident wave train.

When $\alpha \neq 0$, the infinite systems (3.21) and (3.22) have to be solved by truncation. A first approximation to R and T can be obtained by assuming a_1, b_1 to be the only non vanishing coefficients in the systems (3.21) and (3.22), so that we find, after truncation

$$a_1 = \lambda a + \frac{V_1^{(1)} - \pi^2 \lambda a}{\pi^2 + P_{11}^{(1)}}, \tag{3.28}$$

$$b_1 = \mathrm{i}\left(-\lambda a \cos\alpha + \frac{\pi^2 \lambda a \cos\alpha - \mathrm{i}V_1^{(2)}}{\pi^2 + P_{11}^{(4)}}\right). \tag{3.29}$$

The constants $P_{11}^{(1)}$, $P_{11}^{(4)}$, $B_{11}^{(1)}$, $B_{11}^{(2)}$, $V_1^{(1)}$ and $V_1^{(2)}$ can be evaluated by using the result (3.18) and these are given by

$$P_{11}^{(1)} = -\frac{2\pi E_1 I_1(a\lambda)}{\sin\alpha}\left[\left\{K_0(2\gamma h) + \frac{K_1(2\gamma h)}{2\gamma h} - \frac{1}{2}\frac{K_1'(a\gamma)}{I_1'(a\gamma)}\right\}\sin^2\alpha - \frac{\mathrm{d}^2}{\mathrm{d}l^2}F(l,\gamma)|_{l=2h}\right], \tag{3.30}$$

where $I_1(x)$, $K_n(x) (n = 0, 1)$ denote the modified Bessel functions of first kind and second kind respectively and dash denotes its derivative, and

$$E_1 = -2\pi\lambda a I_1'(a\gamma), \tag{3.31}$$

$$F(l,\gamma) = 2\int_0^{\infty} \frac{D(z^2+\gamma^2)^2 + 1 - \epsilon K}{(z^2+\gamma^2)^{\frac{1}{2}}\left\{D(z^2+\gamma^2)^2 + 1 - \epsilon K\right\} - K}\mathrm{e}^{-l\sqrt{z^2+\gamma^2}}\mathrm{d}z, \tag{3.32}$$

the contour being indented below the pole at $z = \mu$,

$$P_{11}^{(4)} = -\frac{2\pi E_1 I_1(a\lambda)}{\sin\alpha}\left[\frac{K_1(2\gamma h)}{2\lambda h}\sin\alpha - \frac{1}{2}\frac{K_1'(a\gamma)}{I_1'(a\gamma)}\sin^2\alpha + \left(\gamma^2 - \frac{\mathrm{d}^2}{\mathrm{d}l^2}\right)F(l,\gamma)|_{l=2h}\right], \tag{3.33}$$

$$B_{11}^{(1)} = P_{11}^{(1)}, \quad B_{11}^{(2)} = -\mathrm{i}\cos\alpha P_{11}^{(4)},$$

$$V_1^{(1)} = \frac{2\pi^2 I_1(a\gamma)}{\sin\alpha}, \tag{3.34}$$

$$V_1^{(2)} = -\mathrm{i}\frac{2\pi^2 I_1(a\gamma)}{\tan\alpha}. \tag{3.35}$$

Now $F(l,\gamma)$ can be simplified as

$$F(l,\gamma) = 2\pi\mathrm{i}\left[\frac{\mathrm{e}^{-l\lambda}}{\cos\alpha}g(\lambda) + \frac{\lambda_1\mathrm{e}^{-l\lambda_1}}{(\lambda_1^2-\gamma^2)^{\frac{1}{2}}}g(\lambda_1) + \frac{\overline{\lambda_1}\mathrm{e}^{-l\overline{\lambda_1}}}{(\overline{\lambda_1}^2-\gamma^2)^{\frac{1}{2}}}g(\overline{\lambda_1})\right] + 2K_0(l\gamma) - 2\left[g(\lambda)M(\lambda) - g(\lambda_1)M(\lambda_1) + g(\overline{\lambda_1})M(\overline{\lambda_1}) - g(\lambda_2)M(\lambda_2) + g(\overline{\lambda_2})M(\overline{\lambda_2})\right], \tag{3.36}$$

with

$$M(u) = \left[\frac{u}{(u^2-\gamma^2)^{\frac{1}{2}}}\ln\left\{\frac{u+(u^2-\gamma^2)^{\frac{1}{2}}}{\gamma}\right\} + \int_0^{lu}\mathrm{e}^x K_0\left(\frac{\gamma x}{u}\right)\mathrm{d}x\right]\mathrm{e}^{-lu}. \tag{3.37}$$

Thus a first order approximation to reflection and transmission coefficients is given by

$$R^{(1)} = -\mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[E_0 + \left(\frac{V_1^{(1)}-\pi^2\lambda a}{\pi^2+P_{11}^{(1)}}\right)E_1 - \left(\frac{\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}}{\pi^2+P_{11}^{(4)}}\right)H_1\right], \tag{3.38}$$

where

$$\begin{aligned} &E_0 = 2\pi\gamma a I_1(a\gamma),\\ &H_1 = 2\pi\mu a I_1'(a\gamma),\\ &T^{(1)} = 1 - \mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[E_0 + \left(\frac{V_1^{(1)}-\pi^2\lambda a}{\pi^2+P_{11}^{(1)}}\right)E_1 + \left(\frac{\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}}{\pi^2+P_{11}^{(4)}}\right)H_1\right]. \end{aligned} \tag{3.39}$$

A second approximation to the reflection co-efficient $R^{(2)}$ and transmission co-efficient $T^{(2)}$ follow from the assumption that a_1, a_2 and b_1, b_2 be the only non vanishing coefficients in the linear systems (3.21) and (3.22). Hence we find from (3.21) and (3.22) that

$$\pi^2 a_s + a_1 P_{1s}^{(1)} + a_2 P_{2s}^{(1)} - \lambda a B_{1s}^{(1)} + \frac{(\lambda a)^2}{2}B_{2s}^{(1)} = V_s^{(1)}, \quad s = 1,2 \tag{3.40}$$

$$\pi^2 b_s + b_1 P_{1s}^{(4)} + b_2 P_{2s}^{(4)} - \lambda a B_{1s}^{(2)} + \frac{(\lambda a)^2}{2}B_{2s}^{(2)} = V_s^{(2)}, \quad s = 1,2 \tag{3.41}$$

From two equations (3.40) and (3.41), we get

$$a_1 = \lambda a + \frac{\left(V_1^{(1)}-\pi^2\lambda a\right)\left(P_{22}^{(1)}+\pi^2\right) - \left(V_2^{(1)} + \frac{\pi^2(\lambda a)^2}{4}(1+\cos^2\alpha)\right)P_{12}^{(1)}}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right) - \left(P_{12}^{(1)}\right)^2}, \tag{3.42}$$

$$a_2 = -\frac{(\lambda a)^2}{4}(1+\cos^2\alpha) + \frac{\left(V_2^{(1)} + \frac{\pi^2(\lambda a)^2}{4}(1+\cos^2\alpha)\right)\left(\pi^2+P_{11}^{(1)}\right) - \left(V_1^{(1)}-\pi^2\lambda a\right)P_{12}^{(1)}}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right) - \left(P_{12}^{(1)}\right)^2} \tag{3.43}$$

and

$$b_1 = \mathrm{i}\left\{-\lambda a\cos\alpha + \frac{\left(\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}\right)\left(\pi^2+P_{22}^{(4)}\right) + \left(\frac{\pi^2(\lambda a)^2}{2}\cos\alpha + \mathrm{i}V_2^{(2)}\right)P_{12}^{(4)}}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right) - \left(P_{12}^{(4)}\right)^2}\right\}, \tag{3.44}$$

$$b_2 = \mathrm{i}\left\{\frac{(\lambda a)^2}{2}\cos\alpha - \frac{\left(\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}\right)P_{12}^{(4)} + \left(\frac{\pi^2(\lambda a)^2}{2}\cos\alpha + \mathrm{i}V_2^{(2)}\right)\left(\pi^2+P_{11}^{(4)}\right)}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right) - \left(P_{12}^{(4)}\right)^2}\right\}, \tag{3.45}$$

The constants can be calculated by using the result (3.18) and these are given as follows:

$$P_{12}^{(1)} = P_{21}^{(1)} = -4\pi\frac{E_1 I_2(a\gamma)}{\sin^2\alpha}\left[-\frac{1}{\lambda^3}\int_\gamma^\infty \frac{z^3}{(z^2-\gamma^2)^{\frac{1}{2}}}\mathrm{d}z + K_1''(2\gamma h)\sin^3\alpha + \frac{\mathrm{d}^3}{\mathrm{d}l^3}F(l,\gamma)|_{l=2h}\right], \tag{3.46}$$

where $I_2(x)$ denotes the modified Bessel function of first kind,

$$P_{22}^{(1)} = 2\pi \frac{E_2 I_2(a\gamma)}{(2-\sin^2\alpha)\sin^2\alpha}\left[4\left\{\frac{1}{\lambda^4}\int_\gamma^\infty \frac{z^4}{(z^2-\gamma^2)^{\frac{1}{2}}}\mathrm{d}z + K_1'''(2\gamma h)\sin^4\alpha + \frac{\mathrm{d}^4}{\mathrm{d}l^4}F(l,\gamma)|_{l=2h}\right\} + \frac{2\sin^3\alpha}{\pi E_1 I_1(a\gamma)}P_{11}^{(1)}\right] \tag{3.47}$$

with

$$E_2 = \frac{2\pi\lambda a}{\sin\alpha}(2-\sin^2\alpha)I_2'(a\gamma), \tag{3.48}$$

$$B_{12}^{(1)} = P_{12}^{(1)}, \quad B_{21}^{(1)} = \frac{1}{2}(1+\cos^2\alpha)P_{12}^{(1)}, \tag{3.49}$$

$$P_{12}^{(4)} = P_{21}^{(4)} = -4\pi\frac{E_1 I_2(a\gamma)}{\sin^2\alpha}\left[\frac{1}{\lambda^3}\int_\gamma^\infty z(z^2-\gamma^2)^{\frac{1}{2}}\mathrm{d}z + K_1''(2\gamma)\sin^3\alpha + \left(\frac{\mathrm{d}^3}{\mathrm{d}l^3} - \sin^2\alpha\frac{\mathrm{d}}{\mathrm{d}l}\right)F(l,\gamma)|_{l=2h}\right], \tag{3.50}$$

$$P_{22}^{(4)} = 8\pi\frac{E_1 I_2(a\gamma)}{(2-\sin^2\alpha)\sin^2\alpha}\left[-\frac{1}{\lambda^4}\int_\gamma^\infty z^2(z^2-\gamma^2)^{\frac{1}{2}}\mathrm{d}z + K_1'''(2\gamma h)\sin^4\alpha + \left(\frac{\mathrm{d}^4}{\mathrm{d}l^4} - \sin^2\alpha\frac{\mathrm{d}^2}{\mathrm{d}l^2}\right)F(l,\gamma)|_{l=2h}\right], \tag{3.51}$$

$$B_{12}^{(2)} = B_{21}^{(2)} = -\mathrm{i}\cos\alpha P_{12}^{(4)}, \tag{3.52}$$

$$B_{22}^{(2)} = -\mathrm{i}\cos\alpha P_{22}^{(4)}, \tag{3.53}$$

$$V_2^{(1)} = -2\pi^2\frac{I_2(a\gamma)}{\sin^2\alpha}(2-\sin^2\alpha), \tag{3.54}$$

$$V_2^{(2)} = 4\pi^2\mathrm{i}\frac{I_2(a\gamma)}{\sin^2\alpha}\cos\alpha. \tag{3.55}$$

Thus the second order approximation to reflection and transmission coefficients is given by

$$R^{(2)} = -\mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\left[\left(1+\frac{(\lambda a)^2}{4}\sin^2\alpha\right)E_0 + \left\{\frac{\left\{V_2^{(1)} + \pi^2\frac{(\lambda a)^2}{4}(1+\cos^2\alpha)\right\}\left[\left(\pi^2+P_{11}^{(1)}\right)E_2 - P_{12}^{(1)}E_1\right]}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right) - \left(P_{12}^{(1)}\right)^2} + \frac{\left(V_1^{(1)} - \pi^2\lambda a\right)\left[\left(P_{22}^{(1)}+\pi^2\right)E_1 - P_{12}^{(1)}E_2\right]}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right) - \left(P_{12}^{(1)}\right)^2}\right\} - \left\{\frac{\left(\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}\right)\left[\left(\pi^2+P_{22}^{(4)}\right)H_1 - P_{12}^{(4)}H_2\right]}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right) - \left(P_{12}^{(4)}\right)^2} + \frac{\left\{\pi^2\frac{(\lambda a)^2}{2}\cos\alpha + \mathrm{i}V_2^{(2)}\right\}\left[P_{12}^{(4)}H_1 - \left(\pi^2+P_{11}^{(4)}\right)H_2\right]}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right) - \left(P_{12}^{(4)}\right)^2}\right\}\right] \tag{3.56}$$

and

$$\begin{aligned}T^{(2)} = 1 - \mathrm{i}\frac{g(\lambda)}{\cos\alpha}\mathrm{e}^{-2\lambda h}\Bigg[&\left(1+\frac{(\lambda a)^2}{4}\sin^2\alpha\right)E_0 + \Bigg\{\frac{\left\{V_2^{(1)}+\pi^2\frac{(\lambda a)^2}{4}(1+\cos^2\alpha)\right\}\left[\left(\pi^2+P_{11}^{(1)}\right)E_2 - P_{12}^{(1)}E_1\right]}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right)-\left(P_{12}^{(1)}\right)^2}\\
&+\frac{\left(V_1^{(1)}-\pi^2\lambda a\right)\left[\left(P_{22}^{(1)}+\pi^2\right)E_1 - P_{12}^{(1)}E_2\right]}{\left(\pi^2+P_{11}^{(1)}\right)\left(\pi^2+P_{22}^{(1)}\right)-\left(P_{12}^{(1)}\right)^2}\Bigg\} + \Bigg\{\frac{\left(\pi^2\lambda a\cos\alpha - \mathrm{i}V_1^{(2)}\right)\left[\left(\pi^2+P_{22}^{(4)}\right)H_1 - P_{12}^{(4)}H_2\right]}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right)-\left(P_{12}^{(4)}\right)^2}\\
&+\frac{\left\{\pi^2\frac{(\lambda a)^2}{2}\cos\alpha + \mathrm{i}V_2^{(2)}\right\}\left[P_{12}^{(4)}H_1 - \left(\pi^2+P_{11}^{(4)}\right)H_2\right]}{\left(\pi^2+P_{11}^{(4)}\right)\left(\pi^2+P_{22}^{(4)}\right)-\left(P_{12}^{(4)}\right)^2}\Bigg\}\Bigg] \qquad (3.57)\end{aligned}$$

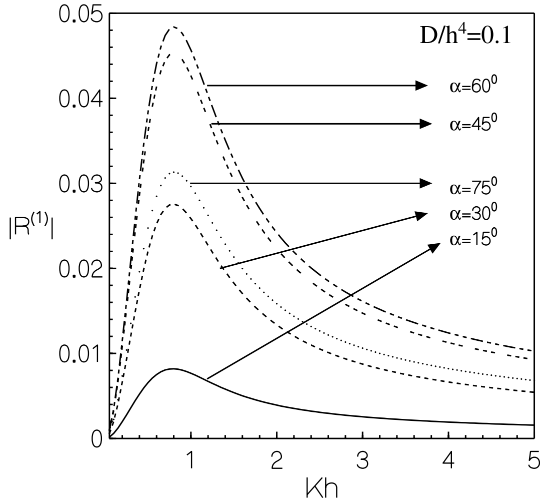

Fig. 1. Reflection coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

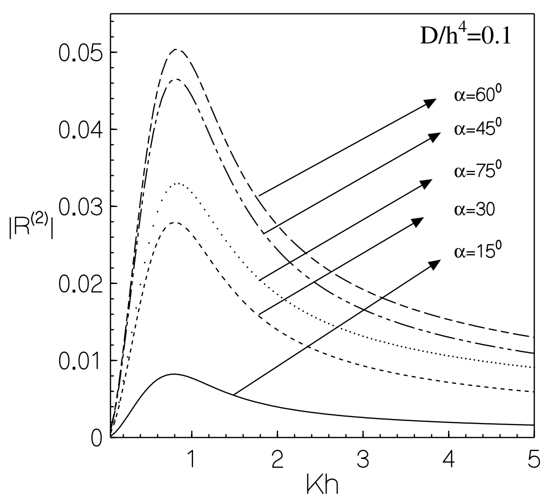

Fig. 2. Reflection coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

with

$$H_2 = -\frac{4\pi\lambda a}{\sin\alpha}\cos\alpha I_2'(a\gamma). \tag{3.58}$$

Also higher order approximations to R and T can calculated, but this is not pursued here.

4. Numerical results

Figs. 1–4 depict the reflection and transmission coefficients against the wave number Kh for $\frac{a}{h}=0.6, \frac{\epsilon}{h}=0.01, \frac{D}{h^4}=0.1$, $|R^{(1)}|$, $|T^{(1)}|$ denoting first order approximations while $|R^{(2)}|$, $|T^{(2)}|$ denoting the second order approximations. The different curves correspond to $\alpha = 15°$, 30°, 45°, 60°, 75°. It is remarkable that the first and second approximations are somewhat close and possess the same features.

Figs. 5–8 show $|R^{(1)}|$, $|T^{(1)}|$ and $|R^{(2)}|$, $|T^{(2)}|$ against Kh for $\frac{a}{h}=0.6, \frac{\epsilon}{h}=0.01, \alpha=60^0$. The different curves correspond to different values of the flexural rigidity, i.e., $\frac{D}{h^4}=0.01, 0.05, 0.1, 0.5, 1$.

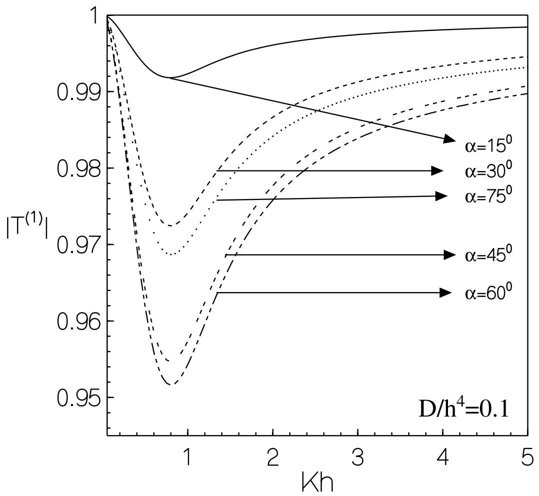

Fig. 3. Transmission coefficient against wave number ($\epsilon/h = 0.01, a/h = 0.6$).

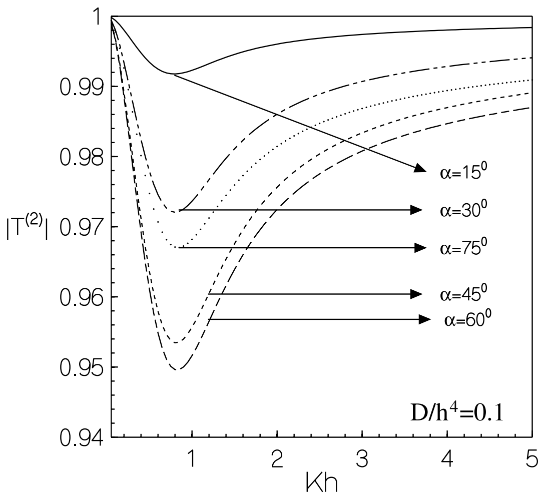

Fig. 4. Transmission coefficient against wave number ($\epsilon/h = 0.01, a/h = 0.6$).

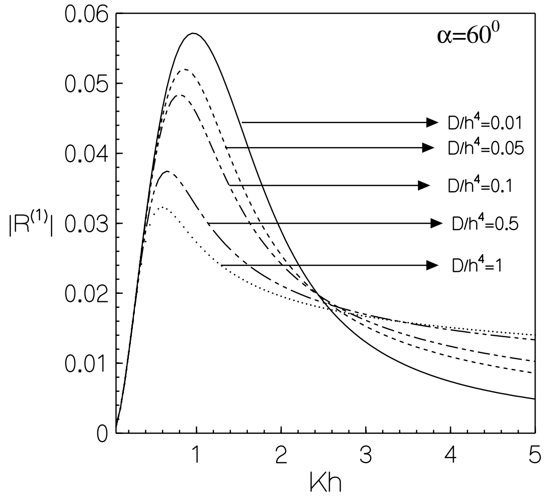

Fig. 5. Reflection coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

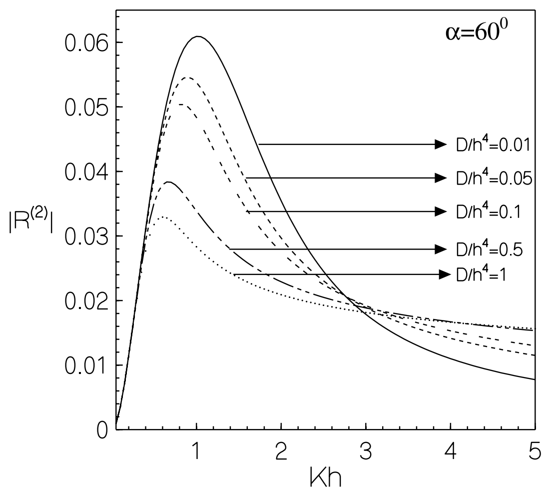

Fig. 6. Reflection coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

The Figs. 1 and 2 show that the reflection coefficient ($|R^{(1)}|$ or $|R^{(2)}|$) regarded as a function of the wave number Kh, first increases as Kh increases, attains a maximum value and then decreases as Kh increases. Also $|R|$ (i.e., $|R^{(1)}|$ or $|R^{(2)}|$) first increases as α increases, attains a maximum value and then decreases as α further increases. Similarly Figs. 3 and 4 show that $|T|$ (i.e., $|T^{(1)}|$ or $|T^{(2)}|$) first decreases as Kh increases, attains a minimum value then increases as Kh further increases. It first decreases as α increases, attains a minimum value and then increases as α further increases. Figs. 5 and 6 show that $|R|$ first decreases as $\frac{D}{h^4}$ increases for low to moderate values of Kh but it increases as $\frac{D}{h^4}$ further increases for somewhat large value of Kh. Figs. 7 and 8 describe the behaviour of $|T|$ which is complimentarily to the behaviour of $|R|$ described by Figs. 5 and 6. This is obvious due to the energy identity $|R|^2 + |T|^2 = 1$.

5. Conclusion

The classical problem of water wave scattering by a long horizontal circular cylinder submerged in deep water beneath a free surface is extended here when the free surface is replaced by a thin ice-cover modelled as a thin elastic plate. Here also, for normal incidence of a wave train, zero reflection occurs for any radius

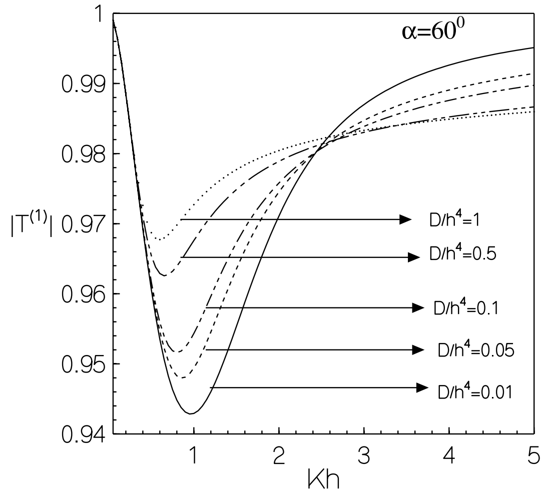

Fig. 7. Transmission coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

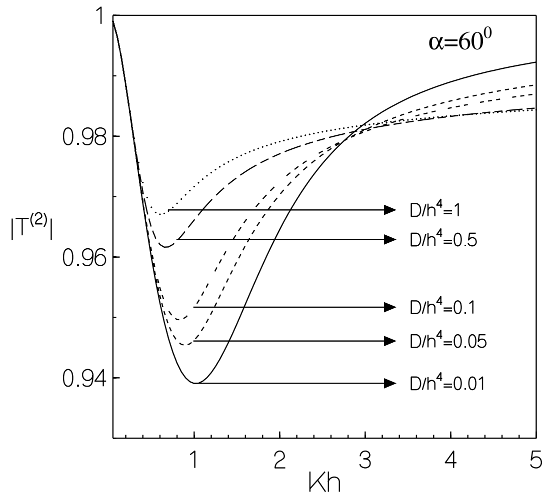

Fig. 8. Transmission coefficient against wave number ($\varepsilon/h = 0.01, a/h = 0.6$).

of the cross-section of the cylinder, submergence depth and wave number. However, for oblique incidence of the wave train, reflection by the submerged cylinder indeed occurs. First and second order approximations to the reflection and transmission coefficients are obtained in terms of computable integrals and depicted graphically against the wave number in a number of figures. For small angle of incidence, the reflection coefficient is seen to be negligible, which is expected. The numerical results reveal that the first and second order approximations are somewhat close, and as such further higher order approximation appears to be not necessary.

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Appendix A

The Eqs. (3.21) and (3.22) can be written in the forms

$$(\mathbf{I}+\mathbf{C}^{(1)})\mathbf{a}+\mathbf{B}^{(1)}=\mathbf{w}^{(1)}, \tag{A.1}$$

$$(\mathbf{I}+\mathbf{C}^{(2)})\mathbf{b}+\mathbf{B}^{(2)}=\mathbf{w}^{(2)}, \tag{A.2}$$

where $\mathbf{I}$ is the infinite unit matrix, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{w}^{(1)}$, $\mathbf{w}^{(2)}$ are the column vectors (a_s), (b_s), $(w_s^{(1)})$, $(w_s^{(2)})$, respectively, where

$$w_s^{(1),(2)}=\frac{1}{\pi^2}V_s^{(1),(2)} \tag{A.3}$$

and the elements in the sth row and nth column of the matrices $\mathbf{C}^{(1)}$, $\mathbf{C}^{(2)}$, $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ are respectively

$$C_{s,n}^{(1)}=\frac{1}{\pi^2}P_{ns}^{(1)}, \tag{A.4}$$

$$C_{s,n}^{(2)}=\frac{1}{\pi^2}P_{ns}^{(4)}, \tag{A.5}$$

$$B_{s,n}^{(1)}=(-1)^n\frac{1}{\pi^2}\frac{(\lambda a)^n}{n!}B_{ns}^{(1)}, \tag{A.6}$$

$$B_{s,n}^{(2)}=(-1)^n\frac{1}{\pi^2}\frac{(\lambda a)^n}{n!}B_{ns}^{(2)}. \tag{A.7}$$

Equations of the forms (A.1) and (A.2) have been studied by Ursell [2].

The linear system (A.1) possesses unique solution under the following conditions:

$$(a1)\quad \sum_{s=1}^{\infty}\left|w_s^{(1)}\right|<\infty, \tag{A.8}$$

$$(a2)\quad \sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\left|C_{s,n}^{(1)}\right|<\infty, \tag{A.9}$$

$$(a3)\quad \sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\left|B_{s,n}^{(1)}\right|<\infty. \tag{A.10}$$

Similarly for the system (A.2):

$$(b1)\quad \sum_{s=1}^{\infty}\left|w_s^{(2)}\right|<\infty, \tag{A.11}$$

$$(b2)\quad \sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\left|C_{s,n}^{(2)}\right|<\infty, \tag{A.12}$$

$$(b3)\quad \sum_{s=1}^{\infty}\sum_{n=1}^{\infty}\left|B_{s,n}^{(2)}\right|<\infty. \tag{A.13}$$

Now

$$\left|w_s^{(1)}\right|=\frac{1}{\pi^2}\left|V_s^{(1)}\right|=2\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}a\right)^{2k+s}(\gamma)^{2k}\lambda^s}{k!(k+s)!}. \tag{A.14}$$

Since the above series is convergent. The condition $(a1)$ is satisfied. Similarly the condition $(b1)$ is satisfied, where

$$\left|w_s^{(2)}\right|=\left|2i\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}a\right)^{2k+s}(\gamma)^{2k}\lambda^{s-1}\mu}{k!(k+s)!}\right|. \tag{A.15}$$

Again

$$\left|C_{s,n}^{(1)}\right|=\frac{1}{\pi^2}\left|P_{ns}^{(1)}\right|=\left|\frac{2^{n+s}aI_n'(a\gamma)I_s(a\gamma)}{\gamma^{n+s-2}}\frac{\partial^{n+s}}{\partial y^n\partial\eta^s}G(x,y;\xi,\eta)|_{x=\xi=0,y=\eta=0}\right|\leqslant|N_1||M_1|, \tag{A.16}$$

where

$$N_1 = 2\left(\frac{a}{h}\right)^n\left(\frac{a}{h}\right)^s\left(\frac{1}{2(n-1)!}+1\right)\left(\sum_{i=1}^{\infty}\frac{\left(\frac{1}{4}a^2\gamma^2\right)^i}{i!(i+n)!}\right)\left(\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}a^2\gamma^2\right)^j}{j!(j+n)!}\right), \tag{A.17}$$

$$M_1 = \int_{\gamma}^{\infty}\frac{u(D_1u^4+1-\epsilon K)+Kh}{\left(u^2-(\gamma h)^2\right)^{\frac{1}{2}}\{u(D_1u^4+1-\epsilon K)-Kh\}}u^{n+s}\mathrm{e}^{-2u}\mathrm{d}u + 2\pi\mathrm{i}\frac{g(\lambda h)}{\cos\alpha}(\lambda h)^{n+s}\mathrm{e}^{-2\lambda h} \tag{A.18}$$

with $D_1 = \frac{D}{h^4}$. Also

$$|C_{s,n}^{(2)}| \leqslant |N_1||M_2|, \tag{A.19}$$

where

$$M_2 = \int_{\gamma}^{\infty}\frac{\left(u^2-(\gamma h)^2\right)^{\frac{1}{2}}\{u(D_1u^4+1-\epsilon K)+Kh\}}{\{u(D_1u^4+1-\epsilon K)-Kh\}}u^{n+s-2}\mathrm{e}^{-2u}\mathrm{d}u + 2\pi\mathrm{i}g(\lambda h)h^2(\lambda h)^{n+s-2}\cos\alpha\mathrm{e}^{-2\lambda h}. \tag{A.20}$$

Since $a < h$ and $|M_1|$ and $|M_2|$ are bounded. Thus the conditions ($a2$) and ($b2$) are satisfied. Again

$$\left|A_{s,n}^{(1),(2)}\right| = \frac{1}{\pi^2}\left|\frac{(\lambda a)^n}{n!}B_{ns}^{(1),(2)}\right|.$$

We see that $B_{ns}^{(1),(2)}$ is the finite series of $P_{ns}^{(1),(4)}$. Hence the conditions ($a3$) and ($b3$) are satisfied. Thus the infinite linear systems (A.1) and (A.2) possess unique solution.

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