THE LANDAU–KOLMOGOROV INEQUALITY REVISITED

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ABSTRACT. We consider the Landau–Kolmogorov problem on a finite interval which is to find an exact bound for \( \| f^{(k)} \| \), for \( 0 < k < n \), given bounds \( \| f \| \leq 1 \) and \( \| f^{(n)} \| \leq \sigma \), with \( \| \cdot \| \) being the max-norm on \([-1, 1]\). In 1972, Karlin conjectured that this bound is attained at the end-point of the interval by a certain Zolotarev polynomial or spline, and that was proved for a number of particular values of \( n \) or \( \sigma \). Here, we provide a complete proof of this conjecture in the polynomial case, i.e. for \( 0 \leq \sigma \leq \sigma_n := \| T_n \| \), where \( T_n \) is the Chebyshev polynomial of degree \( n \). In addition, we prove a certain Schur-type estimate which is of independent interest.

1. Introduction. The Landau–Kolmogorov (LK-) problem consists of finding the upper bound \( M_k \) for the norm of intermediate derivative \( \| f^{(k)} \| \), when the bounds \( \| f \| \leq M_0 \) and \( \| f^{(n)} \| \leq M_n \), for the norms of the function and of its higher derivative, are given.

Here, following our earlier interest [13], we consider the case of a finite interval when \( f \in W_n^\infty [-1, 1] \) and all the norms are the max-norms, \( \| \cdot \| = \| \cdot \|_{L^\infty [-1, 1]} \). Precisely, given \( n, k \in \mathbb{N} \) and \( \sigma \geq 0 \), we define the functional class

\[
W_n^\infty (\sigma) := \{ f : f^{(n-1)} \text{ is abs. cont., } \| f \| \leq 1, \| f^{(n)} \| \leq \sigma \}
\]

and consider the problem of finding the values

\[
M_k(\sigma) := \sup_{f \in W_n^\infty (\sigma)} \| f^{(k)} \|, \quad M_k(x, \sigma) := \sup_{f \in W_n^\infty (\sigma)} | f^{(k)} (x) |, \quad x \in [-1, 1],
\]

The first value (1.1) is related to what is called the pointwise Landau-Kolmogorov inequality (see [9]), and while we are primarily interested in bounding the uniform norm \( \| f^{(k)} \| \) in (1.2), there is no way of doing it other than bounding \( | f^{(k)} (x) | \) pointwise for each \( x \in [-1, 1] \). Note that, for \( \sigma = \sigma_0 = 0 \), we obtain \( W_n^\infty (\sigma_0) = P_{n-1} \), and solution in this case is given by the classic Markov inequality for derivatives of an algebraic polynomial.

Our interest in the case (1.2), when all three norms are the max-norms, is motivated by the fact that there are good chances to add this case to a short list of Landau–Kolmogorov inequalities where a complete solution exists, i.e., a solution that covers all values of \( n, k \in \mathbb{N} \) (and, for a finite interval, all values of \( \sigma \geq 0 \)). The main guideline in finding out how good these chances are is the following conjecture.

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Conjecture 1.1 (Karlin [5]). For all $n, k \in \mathbb{N}$ and all $\sigma > 0$, 
\begin{equation}
\sup_{x \in [-1,1]} m_k(x, \sigma) = m_k(1, \sigma),
\end{equation}
i.e., the function $m_k(\cdot, \sigma)$ in (1.1) reaches its maximal value at the end-point of the interval.

If (1.3) is true for particular set $\{n, k, \sigma\}$, then the function $f \in W_{\infty}^n(\sigma)$ that provides extremum $M_k(\sigma)$ to the value $\|f^{(k)}\|$ over $W_{\infty}^n(\sigma)$ is the same as solution to the pointwise problem (1.1) for the end-point $x = 1$ of the interval $[-1,1]$. The latter solution is, however, known [5] to be a certain Chebyshev or Zolotarev spline $Z_n(\cdot, \sigma)$ (which is just a polynomial for small $\sigma$, see Sect. 3 for further details), i.e.,
\begin{equation}
m_k(1, \sigma) = Z_n^{(k)}(1, \sigma),
\end{equation}
and thus we have a characterization of the extremal function and, at least implicitly, the extremal value of $\|f^{(k)}\|$.

Corollary 1.2. If equality (1.3) is valid for particular $\{n, k, \sigma\}$, then for that set of parameters we have
\begin{equation}
M_k(\sigma) = Z_n^{(k)}(1, \sigma).
\end{equation}

So far, Karlin’s conjecture (1.3) has been proved for small $n$ with all $\sigma$, and for all $n$ with particular $\sigma$, namely in the following cases:
\begin{itemize}
\item $n \in \mathbb{N}$, $\sigma = \sigma_0$, V. Markov, see [15];
\item $n = 2$, all $\sigma$, Chui–Smith [1] ($\sigma \leq \sigma_n$), Landau [6] ($\sigma > \sigma_n$);
\item $n = 3$, all $\sigma$, Sato [10], Zvyagintsev–Lepin [17];
\item $n = 4$, all $\sigma$, Zvyagintsev [16] ($\sigma \leq \sigma_n$), Naidenov [8] ($\sigma > \sigma_n$);
\item $n \in \mathbb{N}$, $\sigma = \sigma_n$, Eriksson [3].
\end{itemize}

Here
\begin{equation}
\sigma_0 := \|T_n^{(n)}\| = 0, \quad \sigma_n := \|T_n^{(n)}\| = 2^{n-1}n!,
\end{equation}
where $T_n(x) := \cos n \arccos x$ is the Chebyshev polynomial of degree $n$ on the interval $[-1,1]$. (Some further qualitative and related results can be found in our survey [13].)

The value $\sigma = \sigma_n$ serves as a borderline between two types of the extremal functions $Z_n(\cdot, \sigma)$:
\begin{itemize}
\item a) polynomial case: if $\sigma \leq \sigma_n$, then $Z_n$ is a Zolotarev polynomial of degree $n$,
\item b) spline case: if $\sigma > \sigma_n$, then $Z_n$ is a Zolotarev perfect spline.
\end{itemize}

Zolotarev polynomials are generalizations of the Chebyshev polynomials, and their definition and properties are given in Sect. 3. We are not considering the spline case here, hence we omit details about Zolotarev splines.

1) Main result. Our main result is a complete solution to the polynomial case. We prove that, for $0 \leq \sigma \leq \sigma_n$, the extreme value of the $k$-th derivative of $f \in W_{\infty}^n(\sigma)$ is provided by the corresponding Zolotarev polynomial.

Theorem 1.3. If
\begin{equation}
n \in \mathbb{N}, \quad 1 \leq k \leq n-1, \quad 0 \leq \sigma \leq \sigma_n,
\end{equation}
then Karlin’s conjecture (1.3) is true, hence
\begin{equation}
M_k(\sigma) = Z_n^{(k)}(1, \sigma).
\end{equation}
Theorem 1.4. We have

\[ \frac{1}{n} n - k + \frac{k \sigma}{n \sigma_n} \leq Z_n^{(k)}(1, \sigma) \leq \frac{n - k}{n} + \frac{k \sigma}{n \sigma_n}, \quad 0 \leq \sigma \leq \sigma_n. \]

The proof of (1.6) for \( 1 \leq k \leq n - 2 \), which forms the major part of the paper, is based on comparing an upper bound for the local extrema of the function \( m_k(\cdot, \sigma) \) with a lower bound for the value \( m_k(1, \sigma) \). This technique is not applicable for the value \( k = n - 1 \), so we cover the case \( k = n - 1 \) using different tools.

Our proof covers all previously known results from (1.5) for \( 0 < \sigma \leq \sigma_n \), in particular, we take care to consider the values \( n = 2, 3, 4 \) when appropriate. We cannot claim the case \( \sigma = \sigma_0 = 0 \), i.e., the Markov inequality, since our proof for \( 0 < \sigma \leq \sigma_n \) makes use of V. Markov’s result that the local maxima of the function \( m_k(\cdot, \sigma_0) \) coincide with the local maxima of the polynomial \( T_n^{(k)} \).

2) Schur-type inequality. The key-role in our proof is played by an estimate which may be viewed as an extension to the higher derivatives of Markov-type results of Schur [11] and Erdős-Szegő [2]. This estimate is of independent interest, so here are further details.

According to the well-known Markov inequality, for \( p \in \mathcal{P}_n \), we have

\[ \sup_{\|p\| \leq 1} |p^{(k)}(x)| \leq T_n^{(k)}(1), \quad x \in [-1, 1], \]

and equality is attained for \( x = 1 \) and \( p = T_n \). Consider the problem of finding the maximum of \( |p^{(k)}(x_0)| \) under additional assumption that \( p^{(k+1)}(x_0) = 0 \), i.e., that \( p \) has a local maximum at \( x = x_0 \). For the first derivative, Schur [11] proved that

\[ \sup_{\|p\| \leq 1, p''(x_0) = 0} |p'(x_0)| < \frac{1}{2} T_n'(1), \]

and some further refinements of this result were given by Erdős-Szegő [2]. We obtain inequality of this type for all derivatives of order \( 1 \leq k \leq n - 2 \). (For \( k = n - 1 \) such inequality does not make sense, since the \((n-1)\)-st derivative of a polynomial of degree \( n \) is a linear function that has no local extrema.)

Theorem 1.4. We have

\[ \sup_{\|p\| \leq 1, p^{(k+1)}(x_0) = 0} |p^{(k)}(x_0)| < \frac{1}{k + 1} \frac{n - 1}{n - 1 + k} T_n^{(k)}(1), \quad 1 \leq k \leq n - 2. \quad (1.7) \]

The extreme value in (1.7) is attained by a certain Zolotarev polynomial, and, if we borrow the words Schoenberg said once about cubic splines, “it demonstrates once again the brave behaviour of Zolotarev polynomials under difficult circumstances”.

3) Organization of the paper. For the case \( 1 \leq k \leq n - 2 \), implementation of the main idea of the proof required long and tedious calculations of various constants which are mostly the values of derivatives of certain polynomials at the end-point of the form \( p^{(m)}(1) \). Here, \( m \) can take the values \( k - 1 \) or \( k \) or \( k + 1 \), and \( p \) is the Chebyshev polynomial \( T_n \) or \( T_{n-1} \), or a polynomial of the form \((x^2 - 1)T_n'(x)\) and alike, so the constants are rather bulky.

Therefore, in order not to get lost with technicalities, we describe in the next section main steps of the proof together with the main inequalities. The latter are then derived in the subsequent sections.
Respectively, everywhere in Sects. 2-10 it is assumed that \(1 \leq k \leq n - 2\).
The case \(k = n - 1\) (with a separate proof) is considered at the very end, in Sect. 11.

2. Main ingredients of the proof.

2.1. Main idea: Local extrema of \(m_k(\cdot, \sigma)\) compared to \(m_k(1, \sigma)\). Karlin’s conjecture (1.3) states that the function \(m_k(\cdot, \sigma)\) in (1.1) (which is a positive even function) reaches its maximal value at the end-points of the interval \([-1, 1]\). To establish this fact it is sufficient to check that, at any point \(x_0\) inside the interval \((-1, 1)\) where \(m_k(\cdot, \sigma)\) takes its local maximum, we have

\[
m_k(x_0, \sigma) = m_k(1, \sigma).
\]

If \(f\) is a function from \(W^n_\infty(\sigma)\) that attains a locally maximal value \(m_k(x_0, \sigma)\), and if \(k \leq n - 2\), then clearly

\[
m_k(x_0, \sigma) = |f^{(k)}(1)|, \quad f^{(k+1)}(x_0) = 0,
\]

and it makes sense to introduce the following quantity:

\[
m^*_k(x_0, \sigma) := \sup \{|f^{(k)}(1)| : f \in W^n_\infty(\sigma), f^{(k+1)}(x_0) = 0\}, \quad x_0 \in [-1, 1].
\]

(2.1)

The next statement follows immediately.

Claim 2.1. If, for given \(n \in \mathbb{N}, 1 \leq k \leq n - 2, \) and \(\sigma > 0\), we have

\[
\max_{x_0 \in [0,1]} m^*_k(x_0, \sigma) \leq m_k(1, \sigma),
\]

then Karlin’s conjecture (1.3) is true.

In order to verify inequality (2.2), we need two different estimates.

a) A good lower bound for the end-point value \(m_k(1, \sigma) = \sup \{|f^{(k)}(1)| : f \in W^n_\infty(\sigma)\}\),

\[
1) \quad m_k(1, \sigma) \geq B_{n,k}(\sigma).
\]

(2.3)

b) A good upper bound for \(|f^{(k)}(x_0)|\), where \(f\) is from \(W^n_\infty(\sigma)\) and satisfies \(f^{(k+1)}(x_0) = 0\).

Actually, if \(x = x_0\) stays sufficiently far away from the end-points \(\pm 1\), then a reasonable upper bound for \(|f^{(k)}(x)|\) can be established irrespectively of whether \(f^{(k+1)}(x)\) vanishes or not. Therefore, for the upper bounds for \(|f^{(k)}(x)|\), we will consider two cases

\[
2) \quad \max_{x_0 \in [\omega_k,1]} m^*_k(x_0, \sigma) \leq A^*_{n,k}(\sigma),
\]

\[
3) \quad \max_{x \in [0,\omega_k]} m_k(x, \sigma) \leq A_{n,k}(\sigma),
\]

(2.4)

with appropriately chosen value \(\omega_k\). The proof is completed by checking that

\[
\max\{A^*_n(\sigma), A_{n,k}(\sigma)\} \leq B_{n,k}(\sigma), \quad 1 \leq k \leq n - 2.
\]
2.2. Lower estimates for \( m_k(1, \sigma) \). Clearly, \( m_k(1, \sigma) \) is monotonically increasing with \( \sigma \), therefore, with \( \sigma_0 := 0 \), we have the trivial estimate
\[
m_k(1, \sigma) \geq m_k(1, \sigma_0) = T_{n-1}^{(k)}(1).
\]
However, this estimate is too rough when \( k \approx cn \), so we will use a finer one.

**Proposition 2.2.** We have
\[
m_k(1, \sigma) \geq B_{n,k}(\sigma) := \left( 1 - \frac{\sigma}{\sigma_n} \right) T_{n-1}^{(k)}(1) + \frac{\sigma}{\sigma_n} T_n^{(k)}(1), \quad 0 \leq \sigma \leq \sigma_n. \tag{2.5}
\]

**Proof.** Let us show that, for a fixed \( x \), \( m_k(x, \sigma) \) as a function of \( \sigma \) is concave. For any \( x \in [-1,1] \), and for any \( \sigma' < \sigma'' \), let \( f_1 \) and \( f_2 \) be the functions such that
\[
f_i(x, \sigma_i) = m_k(x, \sigma_i), \quad f_i \in W_\infty^n(\sigma_i), \quad i = 1, 2.
\]
It is clear that, for any \( \sigma \in [\sigma', \sigma''] \), with \( t_\sigma \) such that \( \sigma = (1 - t_\sigma)\sigma' + t_\sigma\sigma'' \), the function \( f_\sigma := (1 - t_\sigma)f_1 + t_\sigma f_2 \) belongs to \( W_\infty^n(\sigma) \), hence we have
\[
m_k(x, \sigma) \geq f_\sigma(x) = (1 - t_\sigma)f_1(x) + t_\sigma f_2(x) = (1 - t_\sigma)m_k(x, \sigma') + t_\sigma m_k(x, \sigma'').
\]
In particular, with \( \sigma' = \sigma_0 := T_{n-1}^{(n)} = 0 \) and \( \sigma'' = \sigma_n := T_n^{(n)} \), we have \( t_\sigma = \frac{\sigma}{\sigma_n} \), hence
\[
m_k(1, \sigma) \geq \left( 1 - \frac{\sigma}{\sigma_n} \right) m_k(1, \sigma_0) + \frac{\sigma}{\sigma_n} m_k(1, \sigma_n),
\]
But \( m_k(1, \sigma_0) = T_{n-1}^{(k)}(1) \) and \( m_k(1, \sigma_n) = T_n^{(k)}(1) \), hence the result. \( \square \)

2.3. Upper estimate for \( m_k^*(x_0, \sigma) \) for \( x_0 \in [\omega_k, 1] \) and \( 1 \leq k \leq n - 2 \).

**Definition 2.3.** We say that a polynomial \( p \in \mathcal{P}_n \) has an \( n \)-alternance in \([-1, 1]\) if there are \( n \) points \( (t_i) \) such that
\[
-1 \leq t_1 < t_2 < \cdots < t_n \leq 1, \quad p(t_i) = (-1)^i.
\]
If, in addition, we have \( \|p\| = 1 \), then we say that \( p \) has \( n \) equioscillations (the latter polynomials form a family of Zolotarev polynomials, see Sect. 3).

Our starting point is a comparison lemma of the kind similar to the one that was used by Matorin [7] in actually proving that \( m_k(1, \sigma_n) = T_n^{(k)}(1) \).

**Lemma 2.4.** Let \( 1 \leq k \leq n - 2 \), and let \( p \in \mathcal{P}_n[-1, 1] \) be a polynomial that satisfies the following conditions:
1) \( p^{(k+1)}(x_0) = 0 \), \hspace{1cm} 2) \( p \) has an \( n \)-alternance in \([-1, 1] \), \hspace{1cm} 3) \( \|p^{(n)}\| \geq \sigma \).
\[
\tag{2.6}
\]
Then, for any \( f \in W_\infty^n[-1, 1] \) and for any \( x_0 \in [-1, 1] \) such that
\]
\[1') \quad f^{(k+1)}(x_0) = 0, \quad 2') \quad \|f\| \leq 1, \quad 3') \quad \|f^{(n)}\| \leq \sigma, \]
we have
\[
|f^{(k)}(x_0)| \leq |p^{(k)}(x_0)|.
\]

**Remark 2.5.** For any \( x_0 \in [-1, 1] \), and any \( \sigma \geq 0 \), a polynomial \( p \in \mathcal{P}_n \) that satisfies conditions (1)-(3) in (2.6) does exist, e.g., a scaled Zolotarev polynomial (see below).
Proof. Assume the contrary, i.e., that $f^{(k)}(x_0) = p^{(k)}(x_0) / \gamma$ with some $\gamma$ such that $|\gamma| < 1$. Then the function $g := \gamma f$ satisfies

$$2'') \; \|g\| < 1, \quad 3'') \; \|g^{(n)}\| < \sigma,$$

and moreover

$$1'') \; g^{(k)}(x_0) = p^{(k)}(x_0), \quad g^{(k+1)}(x_0) = p^{(k+1)}(x_0) = 0.$$

Consider the difference $h = p - g$. By the $n$-alternation property (2) of $p$, since $\|g\| < 1$, the function $h$ has at least $n - 1$ distinct zeros on $[-1, 1]$, hence $H := h^{(k-1)}$ has at least $n - k$ distinct zeros, say, $(t_i)_{i=1}^{n-k}$, strictly inside $(-1, 1)$, and by (1''), we also have $H'(x_0) = H''(x_0) = 0$. Now, we have three cases.

a) $x_0 = t_i$ for some $i$. Then $t_i$ is a triple zero of $H$, hence $H$ has $n - k + 2$ zeros counting multiplicity.

b) $x_0 \in (t_i, t_{i+1})$ for some $i$. Then $H'$ has $n - k - 2$ zeros outside $[t_i, t_{i+1}]$, one double zero at $x_0$, and one more zero on $(t_i, t_{i+1})$.

c) $x_0 \notin [t_1, t_{n-k}]$. Then $H'$ has $n - k - 1$ zeros in $(t_1, t_{n-k})$, and one double zero at $x_0$.

In any case, $H' = h^{(k)}$ has at least $n-k+1$ zeros on $[-1, 1]$ counting multiplicities, therefore

$$h^{(n)}(x_0) \text{ has at least one sign change on } [-1, 1].$$

On the other hand, by (3) and (3''), we have $|g^{(n)}(x)| < \sigma$ and $|p^{(n)}(x)| \equiv \text{const} \geq \sigma$, hence $|h^{(n)}(x)| = |p^{(n)}(x) - g^{(n)}(x)| > 0$ for all $x \in [-1, 1]$, a contradiction. \hfill \Box

Corollary 2.6. We have

$$m^*_k(x_0, \sigma) \leq |p^{(k)}(x_0)|, \quad x_0 \in [-1, 1],$$

where $p$ is any polynomial of degree $n$ that satisfies conditions (1)-(3) in (2.6).

Let $\{Z_n(\cdot, \theta)\}$ be the family of the Zolotarev polynomials having $n$ equi-oscillations and parameterized with respect to the value of its highest derivative $\theta := Z_n^{(n)}(\cdot, \theta)$ (see Sect. 3 for details). Given $x_0$, our choice for $p$ in (2.7) is a Zolotarev polynomial $Z_n(\cdot, \theta_{x_0})$ such that $Z_n^{(k+1)}(x_0, \theta_{x_0}) = 0$, which is scaled (if needed) to satisfy $\|p^{(n)}\| \geq \sigma$. An advantage of choosing such a $p$ is that, for $x_0$ near the end-point 1, the value of $p^{(k)}(x_0)$ can be further bounded in terms of the single Zolotarev polynomial $Z_n(\cdot, \theta_k)$ such that

$$Z_n^{(k+1)}(1, \theta_k) = 0.$$

Namely, as we show in Sects. 3-4,

$$\max_{x_0 \in [\omega_k, 1]} m^*_k(x_0, \sigma) \leq \max \left\{ 1, \frac{\sigma}{|p_{|\omega_k}|} \right\}^{k/n} \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}.$$

Here, $\omega_k$ is the rightmost zero of $T_n^{(k+1)}$.

In Sects. 5-6, we provide the estimates for the values appearing here on the right-hand side and, thus, in Sect. 7, arrive at the following statement.

Proposition 2.7. We have

$$\max_{x_0 \in [\omega_k, 1]} m^*_k(x_0, \sigma) \leq A_{n,k}(\sigma) := \lambda_k T_n^{(k)}(1) \max \left\{ 1, \frac{1}{|\omega_k|^k} \frac{\sigma}{\sigma_n} \right\}^{k/n}, \quad 0 \leq \frac{\sigma}{\sigma_n} \leq 1$$

(2.8)
where
\[ \lambda_k := \frac{1}{k+1} \frac{n-1}{n-1+k} , \quad \eta_k := \frac{n-(k+1)}{2(2n-(k+1))} . \]

2.4. **Upper estimate for** \( m_k(x, \sigma) \) **for** \( x \in [0, \omega_k] \) **and** \( 1 \leq k \leq n-2 \). To estimate the values of \( m_k(\cdot, \sigma) \) inside the interval, we use a technique based on the Lagrange interpolation. For \( f \in W^n_{\infty}(\sigma) \), let \( \ell_\delta \in \mathcal{P}_{n-1} \) be the polynomial of degree \( n-1 \) that interpolates \( f \) on the mesh \( \delta = (t_i)_{i=0}^{n-1} \), which consists of the points of local extrema of \( T_{n-1}(x) \), i.e.,
\[ \ell_\delta(t_i) = f(t_i), \quad (t_i^2 - 1)T_{n-1}'(t_i) = 0. \]

From the identity \( f^{(k)}(x) = \ell_\delta^{(k)}(x) + (f^{(k)}(x) - \ell_\delta^{(k)}(x)) \) it follows that
\[ |f^{(k)}(x)| \leq D_k(x)\|f\|_\delta + \Omega_k(x)\|f^{(n)}\|, \]
where \( \|f\|_\delta := \max_i |f(t_i)| \) and
\[ D_k(x) := \sup_{\|f\|_{\mathcal{P}_{n-1}, 1 \leq 1}} |p_{n-1}^{(k)}(x)|, \quad \Omega_k(x) := \sup_{\|f^{(n)}\| \leq 1} |f^{(k)}(x) - \ell_\delta^{(k)}(x)|. \]

Since \( f \in W^n_{\infty}(\sigma) \) means \( \|f\| \leq 1 \) and \( \|f^{(n)}\| \leq 1 \), we obtain
\[ \max_{x \in [0, \omega_k]} m_k(x, \sigma) \leq \max_{x \in [0, \omega_k]} D_k(x) + \max_{x \in [0, \omega_k]} \Omega_k(x) \cdot \sigma. \]

In Sect. 8, we show that calculation of the maxima on the right-hand side is reduced to computing the maxima of some specific polynomials, and that leads to the following estimate.

**Proposition 2.8.** We have
\[ \max_{x \in [0, \omega_k]} m_k(x, \sigma) \leq A_{n,k}(\sigma) := \frac{3}{2k+1} T_n^{(k)}(1) + \frac{2}{2k+1} \frac{2(k+1)}{n+k} \frac{T_n^{(k)}(1)}{\sigma_n} \cdot \sigma_n. \]

The latter estimate is not particularly good for small \( k \), so in Sect. 9, for \( \sigma = \sigma_n \), we derive another one:
\[ \max_{x \in [0, \omega_k]} m_k(x, \sigma_n) \leq \tilde{A}_{n,k}(\sigma_n) := \left( 1 + \sin \frac{k+1}{2n} \right)^k T_n^{(k)}(\omega_k). \]

2.5. **Final step for the case** \( 1 \leq k \leq n-2 \). The constants \( B, A^* \) and \( A \) in the estimates (2.5), (2.8) and (2.9) are easy to compare (they are simple functions of \( t = \sigma/\sigma_n \)) and, in Sect. 10, we prove that if \( n \in \mathbb{N}, 1 \leq k \leq n-2 \) and \( 0 \leq \sigma \leq \sigma_n \), then
\[ \max \{ A_{n,k}(\sigma), A^*_{n,k}(\sigma) \} \leq B_{n,k}(\sigma), \]
and that implies Theorem 1.3 for \( k \leq n-2 \), namely
\[ \max_{x \in [-1,1]} m_k(x, \sigma) = m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n, \quad 1 \leq k \leq n-2. \]

2.6. **The case** \( k = n-1 \). For \( k = n-1 \), our previous technique of estimating local maxima of \( m_k(x, \sigma) \) using polynomials as in (2.6) is not applicable. However, it is possible to prove that
\[ \sup_{x \in [-1,1]} m_k(x, \sigma) = m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n, \quad k = n-1, \]
using Lagrange interpolation alone. This is done in Sect. 11, and that completes the proof of Theorem 1.3.
3. Zolotarev polynomials. Here, we remind some facts about Zolotarev polynomials taking some extracts from our survey [15, p. 240–242]. Note that we use here a slightly different parametrization for $Z_n$ than in [15].

**Definition 3.1.** A polynomial $Z_n \in \mathcal{P}_n$ is called a Zolotarev polynomial if it has at least $n$ equioscillations on $[-1, 1]$, i.e. if there exist $n$ points 

$$-1 \leq \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n \leq 1$$

such that

$$(-1)^{n-i} Z_n(\tau_i) = \|Z_n\| = 1.$$ 

There are many Zolotarev polynomials, for example the Chebyshev polynomials $T_n$ and $T_{n-1}$ of degree $n$ and $n-1$, with $n+1$ and $n$ equioscillation points, respectively. One needs an additional parameter to get uniqueness. We will use parametrization through the value of the $n$-th derivative of $Z_n$:

$$\|Z_n^{(n)}\| = \theta \iff Z_n(x) := Z_n(x, \theta) := \theta^n x^n + \sum_{i=0}^{n-1} a_i(\theta)x^i.$$ 

By Chebyshev’s result, $\|p^{(n)}\| \leq \|T_n^{(n)}\| \|p\|$, so the range of the parameter is

$$-\sigma_n \leq \theta \leq \sigma_n, \quad \sigma_n = \|T_n^{(n)}\| = 2^{n-1}n!.$$

As $\theta$ traverses the interval $[-\sigma_n, \sigma_n]$, Zolotarev polynomials go through the following transformations:

$$-T_n(x) \rightarrow -T_n(ax + b) \rightarrow Z_n(x, \theta) \rightarrow T_{n-1}(x) \rightarrow Z_n(x, \theta) \rightarrow T_n(cx + d) \rightarrow T_n(x).$$

Zolotarev polynomials subdivide into 3 groups depending on the structure of the set $A := (\tau_i)$ of their equioscillation points.

1) $A$ contains $n+1$ points: then $Z_n$ is the Chebyshev polynomial $T_n$.

2) $A$ contains $n$ points, but only one of the endpoints: then $Z_n$ is a scaled Chebyshev polynomial $T_n(ax + b)$, $|a| < 1$.

3) $A$ contains $n$ points including both endpoints: then $Z_n$ is called a proper Zolotarev polynomial, and it is either of degree $n$, or the Chebyshev polynomial $T_{n-1}$ of degree $n-1$.

For a proper Zolotarev polynomial $Z_n(\cdot, \theta)$, besides the interior alternation points $(\tau_i)_{i=2}^{n-1}$, there is a point $\beta = \beta(\theta)$ outside $[-1, 1]$ where its first derivative vanishes.

V. Markov (see [15]) proved that zeros of $Z_n'(\cdot, \theta)$ are monotonically increasing functions of $\theta \in [-\sigma_n, \sigma_n]$, with $\beta$ going through the infinity as $\theta$ passes the zero. It follows that, for any $\theta_1, \theta_2$, zeros of $Z_n'(\cdot, \theta_1)$ and $Z_n'(\cdot, \theta_2)$ interlace with each other, hence by the Markov interlacing property the same is true for their derivatives of any order. In particular, with $\theta_1 = 0$, i.e., with $Z_n(\cdot, \theta_1) = T_{n-1}$, the following lemma is true.
Lemma 3.2. Let \((\alpha_i^M)_{i=1}^{M-1}\) be the zeros of \(T_{n-1}^{(m)}\) in increasing order, and, for any given \(\theta \neq 0\), let \((\tau_i^M)_{i=1}^{M-1}\) be the zeros of \(Z_n^{(m)}(\cdot, \theta)\). Then, \((\alpha_i)\) and \((\tau_i)\) interlace, i.e.,
\[
\tau_1 < \alpha_1 < \tau_2 < \alpha_2 < \cdots < \alpha_{M-1} < \tau_M.
\]

Another consequence of the interlacing property is the following observation.

Lemma 3.3. Let \(\omega_k\) be the rightmost zero of \(T_n^{(k+1)}\), and let \(Z_n(\cdot, \theta_k)\) be the Zolotarev polynomials whose \((k + 1)\)-st derivative vanishes at \(x = 1\), i.e.,
\[
T_n^{(k+1)}(\omega_k) = 0, \quad Z_n^{(k+1)}(1, \theta_k) = 0.
\]

Further, for a given \(x_0 \in (\omega_k, 1)\), let \(Z_n(\cdot, \theta_{x_0})\) be the Zolotarev polynomial such that
\[
Z_n^{(k+1)}(x_0, \theta_{x_0}) = 0, \quad x_0 \in (\omega_k, 1).
\]

Then
\[
|\theta_k| < |\theta_{x_0}| < \sigma_n.
\]

Proof. According to our parametrization of \(Z_n(\cdot, \theta)\), we have \(Z_n(\cdot, -\sigma_n) = -T_n\) and \(Z_n(\cdot, \sigma_0) = T_{n-1}\), therefore as \(\theta\) increases from \(-\sigma_n\) to \(\sigma_0\), the rightmost zero \(x_0\) of \(Z_n^{(k+1)}(\cdot, \theta)\) increases from \(\omega_k\) to \(+\infty\), passing through the value 1 for some \(\theta := \theta_k\). Hence,
\[
\omega_k < x_0 < 1 \iff -\sigma_n < \theta_{x_0} < \theta_k < 0,
\]
and that proves the statement. \(\square\)

4. An Erdős–Szegő-type result. According to the well-known Markov inequality
\[
\sup_{\|p\| \leq 1} |p'(x)| \leq n^2, \quad x \in [-1, 1],
\]
and equality is attained at \(x = 1\) for \(p = T_n\).

In 1913, Schur [11] considered the problem of finding the maximum of \(|p'(x_0)|\) under additional assumption that \(p''(x_0) = 0\). He proved that
\[
\sup_{\|p\| \leq 1, p''(x_0) = 0} |p'(x_0)| < \frac{1}{2} n^2.
\]

Moreover, he showed that if \(\lambda_n\) is the least constant in front of \(n^2\), then, for \(\lambda_\infty := \limsup_{n \to \infty} \lambda_n\), we have
\[
0.217 \cdots \leq \lambda_\infty \leq 0.465 \cdots.
\]

In 1942, Erdős and Szegő [2] refined Shur’s result by showing that the limit \(\lambda_\infty = \lim_{n \to \infty} \lambda_n\) exists and it is equal to
\[
\lambda_\infty = \kappa^{-2} (1 - E/K)^2 = 0.3124 \cdots
\]
where \(E, K\) are the complete elliptic integrals associated with the modulus \(\kappa\). (They did not improve the uniform bound (4.1) though.)

In between, they showed that, for any \(x_0 \in [-1, 1]\), the supremum of \(|p'(x_0)|\) subject to \(p''(x_0) = 0\) is attained when \(p\) is a Zolotarev polynomial \(Z_n(\cdot, \theta)\), and that for the maximum of the extreme value \(|p'(x_0)|\) over \(x_0 \in [-1, 1]\) we have the sharp estimate
\[
\max_{x_0 \in [-1, 1]} \sup |p'(x_0)| = \max \{|T_n'(\omega_1)|, |Z_n'(1, \theta_1)|\}.
\]
In this section, we extend this result to the derivatives of order \(1 \leq k \leq n-2\), and then use this generalization to give an estimate for the value of \(m_k(x_0, \sigma)\) in terms of the Zolotarev polynomial \(Z_n(\cdot, \theta)\).

Denote by
\[
\mu_k(x) := \sup_{\|p\| \leq 1} |p^{(k)}(x)|, \quad x \in [-1, 1],
\]
the best constant in the pointwise Markov inequality, and by
\[
\mu_k^*(x_0) := \sup_{\|p\| \leq 1, p^{(k+1)}(x_0) = 0} |p^{(k)}(x_0)| \quad x_0 \in [-1, 1], \quad 1 \leq k \leq n-2,
\]
the best constant in the pointwise Schur-type inequality. It is clear that
\[
\mu_k^*(x_0) \leq \mu_k(x_0), \quad x_0 \in [-1, 1],
\]
and that equality occurs only if \(\mu_k^*(x_0) = 0\), i.e., if \(x_0\) is a point of local extremum (maximum or minimum) of the function \(\mu_k(\cdot)\) inside \([-1, 1]\).

The next two lemmas are straightforward extensions to the case \(k \geq 1\) of the arguments given by Erdős-Szegő in [2, pp.461-462] for \(k = 1\).

**Lemma 4.1.** For any \(\theta\), if \(x_0 \in [-1, 1]\) is such that \(Z^{(k+1)}_n(x_0, \theta) = 0\), then
\[
\mu_k^*(x_0) = Z^{(k)}_n(x_0, \theta).
\]
Conversely, for any \(x_0 \in [-1, 1]\), with some \(\theta = \theta_{x_0}\), there is a polynomial \(Z_n(\cdot, \theta)\) such that (4.3) is true.

**Proof.** The first part follows by the arguments similar to those given in the proof of Lemma 2.4. The second part is a consequence of the monotone behaviour of zeros of \(Z_n^{(k+1)}(\cdot, \theta)\) as functions of \(\theta\), which we discussed in Lemmas 3.2-3.3. Precisely, it follows that those zeros cover the entire interval \([-1, 1]\).

**Lemma 4.2.** Let \(x_0 \in [-1, 1]\) be a point such that
\[
\mu_k^*(x_0) < \mu_k(x_0) \quad \text{and} \quad x_0 \neq \pm 1.
\]
Then, for small \(\delta > 0\), there is a point \(x_1 \in [x_0 - \delta, x_0 + \delta]\), such that
\[
\mu_k^*(x_0) < \mu_k^*(x_1).
\]

**Proof.** Let \(\mu_k^*(x_0) = Z^{(k)}_n(x_0)\), where \(Z^{(k+1)}_n(x_0) = 0\), and let \(p \in \mathcal{P}_n\) be a polynomial such that
\[
p^{(k)}(x_0) > Z^{(k)}_n(x_0) > 0, \quad \|p\| = 1.
\]
Then the polynomial \(q_k(\cdot) := (1 - \epsilon)Z_n + \epsilon p\) satisfies
\[
\|q_k(\cdot)\| \leq 1, \quad q_k^{(k)}(x_0) > Z^{(k)}_n(x_0) = \mu_k^*(x_0),
\]
and, for small \(\epsilon\), its \(k\)-th derivative has a local maximum in the neighbourhood of \(x_0\) (because \(Z^{(k)}_n(\cdot)\) has). Let \(x_1\) be the point of that maximum, i.e., \(q_k^{(k+1)}(x_1) = 0\). Then \(q_k^{(k)}(x_1) > q_k^{(k)}(x_0)\), and respectively
\[
\mu_k^*(x_0) < q_k^{(k)}(x_0) < q_k^{(k)}(x_1) \leq \mu_k^*(x_1),
\]
the latter inequality by the definition of \(\mu_k^*(\cdot)\).

**Corollary 4.3.** Let \(\eta\) be a point of local maximum of the function \(\mu_k^*(\cdot)\). Then
\[
\mu_k^*(\eta) = \mu_k(\eta).
\]
Proof. If \( \mu_k^*(\eta) < \mu_k(\eta) \), then by the previous lemma the function \( \mu_k^*(\cdot) \) takes larger values than \( \mu_k^*(\eta) \) in a neighbourhood of \( \eta \), thus \( \eta \) is not a local maximum, a contradiction. \( \square \)

The next theorem is an extension to the case \( k \geq 1 \) of the Erdős-Szegő result (4.2).

**Proposition 4.4.** Let \( \omega_k \) be the rightmost zero of \( T_n^{(k+1)} \), and let \( Z_n(x, \theta_k) \) be the Zolotarev polynomial such that

\[
Z_n^{(k+1)}(1, \theta_k) = 0.
\]

Then

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}.
\] (4.4)

**Proof.** Let \( (\eta_i) \) be the points of local maxima of \( \mu_k^*(\cdot) \) inside the interval \((-1,1)\).

Then

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{|\mu_k^*(\eta_i)|, \mu_k^*(1)\}.
\]

Corollary 4.3 shows that, inside \((-1,1)\), the local maxima of \( \mu_k^*(\cdot) \) coincide with the extrema (maxima or minima) of \( \mu_k(\cdot) \). On the other hand, V. Markov (see [15]) proved that the local maxima of \( \mu_k(\cdot) \) coincide with those of \( |T_n^{(k)}| \). Hence

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{|T_n^{(k)}(\xi_i)|, \mu_k^*(1)\}, \quad \text{where} \quad T_n^{(k+1)}(\xi_i) = 0.
\]

Let us estimate the values on the right-hand side. It is known that the values of local maxima of \( |T_n^{(k)}| \) increase towards the end-points, i.e.,

\[
\max_i |T_n^{(k)}(\xi_i)| = |T_n^{(k)}(\omega_k)|,
\]

where \( \omega_k \) is the rightmost zero of \( |T_n^{(k+1)}| \). Also, by Lemma 4.1,

\[
\mu_k^*(1) = |Z_n^{(k)}(1, \theta_k)|,
\]

and that completes the proof. \( \square \)

We finish this section by using the previous proposition to derive an estimate for the value of \( m_k^*(x_0, \sigma) \) related to our Landau-Kolmogorov problem.

**Proposition 4.5.** Let \( \omega_k \) be the rightmost zero of \( T_n^{(k+1)} \), and let \( Z_n(x, \theta_k) \) be the Zolotarev polynomial such that

\[
Z_n^{(k+1)}(1, \theta_k) = 0.
\]

Then

\[
\max_{x_0 \in [\omega_k,1]} m_k^*(x_0, \sigma) \leq \max \left\{1, \frac{\sigma}{|\theta_k|} \right\}^{k/n} \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}.
\] (4.5)

**Proof.** According to Corollary 2.6,

\[
m_k^*(x_0, \sigma) \leq |p^{(k)}(x_0)|,
\]

where \( p \) is any polynomial of degree \( n \) such that

1) \( p^{(k+1)}(x_0) = 0 \), \quad 2) \( p \) has an \( n \)-alternance in \([-1,1]\), \quad 3) \( \|p^{(n)}\| \geq \sigma \).

We take \( p \) as a scaled Zolotarev polynomial \( Z_n(\cdot, \theta_{x_0}) \) such that \( Z_n^{(k+1)}(x_0, \theta_{x_0}) = 0 \). The latter satisfies conditions (1)-(2) above, and its highest derivative has the value
\( \theta_{x_0} \). So, if \( |\theta_{x_0}| \geq \sigma \), then condition (3) is fulfilled with \( p = Z_n(\cdot, \theta_{x_0}) \), but if \( |\theta_{x_0}| < \sigma \), then we have to scale \( Z_n \) to ensure (3). So we set

\[
p(x) := Z_n(x_0 + \gamma_0^{1/n}(x - x_0), \theta_{x_0}), \quad \gamma_0 := \max \left\{ 1, \frac{\sigma}{|\theta_{x_0}|} \right\},
\]

and obtain

\[
m^*_k(x_0, \sigma) \leq |p^{(k)}(x_0)| = \max \left\{ 1, \frac{\sigma}{|\theta_{x_0}|} \right\}^{k/n} |Z_n^{(k)}(x_0, \theta_{x_0})|. \tag{4.6}
\]

Now, by (4.3)-(4.4), we have

\[
|Z_n^{(k)}(x_0, \theta_{x_0})| = \mu_k^*(x_0) \leq \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\},
\]

and, by Lemma 3.3,

\[
\omega_k \leq x_0 \leq 1 \implies |\theta_k| \leq |\theta_{x_0}| \leq \sigma_n \implies \left| \frac{\sigma}{\theta_{x_0}} \right| \leq \left| \frac{\sigma}{\theta_k} \right|,
\]

so that (4.6) implies (4.5). \( \Box \)

5. Upper estimates for \( T_n^{(k)}(\omega_k) \) and \( Z_n^{(k)}(1, \theta_k) \). First, we give some upper estimates for the values \( T_n^{(k)}(\omega_k) \) relative to the value \( T_n^{(k)}(1) \).

Lemma 5.1. Let \( \omega_k := \omega_{n,k} \) be the rightmost zero of \( T_n^{(k+1)} \). Then

\[
a) \quad |T_n^{(k)}(\omega_k)| \leq \frac{1}{2k+1} T_n^{(k)}(1), \quad n \in \mathbb{N}, \ 1 \leq k \leq n - 1;
\]

\[
b) \quad |T_n'(\omega_1)| \leq \frac{1}{4} T_n'(1), \quad n \geq 5, \ k = 1. \tag{5.1}
\]

Proof. The first inequality was proved by Eriksson [3], the second inequality is due to Erdős–Szegő [2, p. 464]. \( \Box \)

Next, we give some upper estimates for the constant \( \lambda_k := \lambda_{n,k} \) such that

\[
Z_n^{(k)}(1, \theta_k) \leq \lambda_k T_n^{(k)}(1). \tag{5.2}
\]

We get those estimates using the following lemma.

Lemma 5.2. Let \( p \in \mathcal{P}_n \) be any polynomial that satisfies the following conditions:

1) \( p^{(k+1)}(1) = 0 \), \quad 2) \( p \) has an \( n \)-alternance on \([-1, 1]\). \tag{5.3}

If \( Z_n^{(k+1)}(1, \theta_k) = 0 \), then

\[
|Z_n^{(k)}(1, \theta_k)| \leq |p^{(k)}(1)|. \tag{5.4}
\]

Proof. The proof is parallel to the proof of Lemma 2.4, since \( Z_n \) satisfies \( \|Z_n\| \leq 1 \). Assuming the contrary to (5.4), we derive that, with some \( |\gamma| < 1 \), the \( n \)-th derivative of \( h := p - \gamma Z_n \) should change its sign, and that is impossible as \( h \) is a polynomial of degree \( n \). \( \Box \)

1) We start with the simplest construction of \( p \) in (5.3) to give an idea of how an estimate for \( \lambda_k \) can be obtained, and then we improve it a little bit with a more technical work.

Lemma 5.3. We have

\[
|Z_n^{(k)}(1, \theta_k)| \leq \frac{1}{k+1} T_n^{(k)}(1). \tag{5.5}
\]
Proof. Take

\[ p(x) := T_n(x) - cq(x), \quad q(x) := (x - 1)T'_n(x), \quad c := \frac{T_n^{(k+1)}(1)}{q^{(k+1)}(1)}, \]

so that \( p \) has an \( n \)-alternance on \([-\cos \frac{\pi}{n}, 1]\) for any \( c \), and \( p^{(k+1)}(1) = 0 \) by the choice of particular \( c \). Then

\[ p^{(k)}(1) = T_n^{(k)}(1) - cq^{(k)}(1) = \left( 1 - \frac{T_n^{(k+1)}(1)}{q^{(k+1)}(1)} \right) T_n^{(k)}(1), \]

and since \( q^{(m)}(1) = mT_n^{(m)}(1) \), it follows that

\[ p^{(k)}(1) = \left( 1 - \frac{k}{k+1} \right) T_n^{(k)}(1) = \frac{1}{k+1} T_n^{(k)}(1), \]

and lemma is proved.

2) The next statement improves slightly the previous estimate for large \( k \approx cn \).

**Lemma 5.4.** We have

\[ |Z_n^{(k)}(1, \theta_k)| \leq \lambda_k T_n^{(k)}(1), \quad 1 \leq k \leq n - 2, \tag{5.6} \]

where

\[ \lambda_k := \frac{1}{k+1} \frac{n-1}{n-1+k}. \tag{5.7} \]

Proof. Take

\[ p(x) = T_{n-1}(x) - cq(x), \quad q(x) := (x^2 - 1)T'_{n-1}(x), \quad c := \frac{T_{n-1}^{(k+1)}(1)}{q^{(k+1)}(1)}, \]

so \( p \) has an \( n \)-alternance on \([-1, 1]\) and satisfies \( p^{(k+1)}(1) = 0 \). Then

\[ p^{(k)}(1) = T_{n-1}^{(k)}(1) - cq^{(k)}(1) = \left( 1 - \frac{T_{n-1}^{(k+1)}(1)}{q^{(k+1)}(1)} \right) T_{n-1}^{(k)}(1) =: \hat{\lambda}_{n,k} T_{n-1}^{(k)}(1). \]

Since \( q'(x) = (x^2 - 1)T''_{n-1}(x) + 2xT'_{n-1}(x) = xT'_{n-1}(x) + (n-1)^2 T_{n-1}(x) \), we have

\[ q^{(m)}(1) = T_{n-1}^{(m)}(1) + ((n-1)^2 + (m-1))T_{n-1}^{(m-1)}(1), \]

and using

\[ T_{n-1}^{(k+1)}(1) = \frac{(n-1)^2 - k^2}{2k+1} T_{n-1}^{(k)}(1), \quad T_{n-1}^{(k-1)}(1) = \frac{2k-1}{(n-1)^2 - (k-1)^2} T_{n-1}^{(k)}(1), \]

we obtain, after some simplifications,

\[ p^{(k)}(1) = \hat{\lambda}_k T_{n-1}^{(k)}(1), \tag{5.8} \]

where

\[ \hat{\lambda}_k = 1 - \frac{k}{k+1} \frac{(n-1)^2 - k^2}{(n-1)^2 - (k-1)^2} \frac{2(n-1)^2 + (k-1)}{2(n-1)^2 + k}. \]
Further simplifications give

\[
\hat{\lambda}_k = 1 - \frac{k}{k + 1} \frac{(n - 1) - k}{(n - 1) - (k - 1)} \frac{(n - 1) + k}{(n - 1) + (k - 1)} \frac{2(n - 1)^2 + (k - 1)}{2(n - 1)^2 + k}
\]

(\text{\textbullet}) \leq 1 - \frac{k}{k + 1} \frac{(n - 1) - k}{(n - 1) - (k - 1)} \leq \frac{1}{k + 1} \frac{n}{n - k},

where in (\text{\textbullet}) we used \(\frac{n_1 + 1}{n_1} \frac{n_2}{n_2 + 1} > 1\) for \(n_1 < n_2\). Substituting in (5.8) the equality

\[
T_n^{(k)}(1) = \rho_k T_n^{(k)}(1), \quad \rho_k = \frac{n - 1}{n} \frac{n - k}{n - 1 + k},
\]

we obtain

\[
p^{(k)}(1) = \lambda_k T_n^{(k)}(1), \quad \lambda_k = \hat{\lambda}_k \rho_k \leq \frac{1}{k + 1} \frac{n}{n - k} \rho_k = \frac{1}{k + 1} \frac{n}{n - 1 + k},
\]

and that proves (5.6).

\[\square\]

Remark 5.5. For small \(n < 20\), the Zolotarev polynomials \(Z_n(x, \theta_k)\) such that \(Z_n^{(k+1)}(1, \theta_k) = 0\) can be constructed numerically. Those calculations show that

\[|Z_n^{(k)}(1, \theta_k)| = \lambda_k T_n^{(k)}(1), \quad \frac{1}{2k + 2} < \lambda_k < \frac{1}{2k + 1},\]

and that means that our constant (5.7) have the right order with respect to \(k\), and differ from the exact constant \(\lambda_k^\prime\) by the factor 2 at most.

6. Lower bound for \(\theta_k := Z_n^{(n)}(\cdot, \theta_k)\). In order to proceed further with the estimate (4.5)

\[\max_{x_0 \in [\omega_k, 1]} m^*_k(x_0, \sigma) \leq \max\{1, \frac{\sigma}{|\theta_k|}\}^{k/n} \max\{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\},\]

we need a lower bound for \(|\theta_k|\), which is the \(n\)-th derivative of the Zolotarev polynomial \(Z_n^{(n)}(\cdot, \theta_k)\) such that \(Z_n^{(k+1)}(1, \theta_k) = 0\). In this section, for convenience, we assume that the zero value is taken at \(x = -1\), which means that we consider reflection of the polynomial \(Z_n^{(n)}(\cdot, \theta_k)\).

Lemma 6.1. Let \(Z_n(x, \theta_k)\) be a Zolotarev polynomial such that

\[Z_n^{(k+1)}(-1, \theta_k) = 0.\] (6.1)

Then

\[\theta_k := \|Z_n^{(n)}\| \geq \eta_k \sigma_n, \quad \eta_k := \frac{n - (k + 1)}{2(n - (k + 1))}.\] (6.2)

Proof. Set \(m = k + 1\) and \(M = n - m\), and denote by \((\tau_i)_{i=1}^M\) the zeros of \(Z_n^{(m)}(\cdot, \theta_k)\) in increasing order:

\[-1 = \tau_1 < \tau_2 < \cdots < \tau_M < 1,\]

so that the first zero is \(\tau_1 = -1\) in accordance with (6.1). Then

\[Z_n^{(m)}(x) = A(x + 1)(x - \tau_2) \cdots (x - \tau_M),\]

where

\[A = \frac{Z_n^{(m)}(1)}{2(1 - \tau_2) \cdots (1 - \tau_M)} = \frac{1}{2} \frac{A_1}{A_2},\] (6.3)
and respectively
\[ |\theta_k| = \|Z_n^{(m)}\| = AM = \frac{M!A_1}{2A_2}. \] (6.4)

Let us find lower bounds for the constants \(A_1\) and \(1/A_2\) defined in (6.3).

1) Let \((\alpha_i)_{i=1}^{M-1}\) be the zeros of \(T_{n-1}^{(m)}\) in increasing order. By Lemma 3.2, they interlace with zeros of \(Z_n^{(m)}\), i.e.
\[-1 = \tau_1 < \alpha_1 < \tau_2 < \alpha_2 < \cdots < \alpha_{M-1} < \tau_M < 1,\]
therefore
\[ \frac{1}{A_2} := \frac{1}{(1-\tau_2)\cdots(1-\tau_M)} > \frac{1}{(1-\alpha_1)\cdots(1-\alpha_{M-1})}. \]

On the other hand,
\[ T_{n-1}^{(m)}(x) = \frac{\|T_{n-1}^{(m-1)}\|}{(M-1)!}(x-\alpha_1)\cdots(x-\alpha_{M-1}), \]
hence
\[ T_{n-1}^{(m)}(1) = \frac{\|T_{n-1}^{(m-1)}\|}{(M-1)!}(1-\alpha_1)\cdots(1-\alpha_{M-1}), \]
and respectively
\[ \frac{1}{A_2} > \frac{1}{(M-1)!} \frac{\|T_{n-1}^{(m-1)}\|}{T_{n-1}^{(m)}(1)}. \] (6.5)

2) The lower bound for \(A_1 := Z_n^{(m)}(1, \theta_k)\) is provided by Proposition 2.2, because Zolotarev polynomials \(Z_n(\cdot, \sigma)\) give the value of \(m_k(1, \sigma) = Z_n^{(m)}(1, \sigma)\) for any \(k\) and any \(0 \leq \sigma \leq \sigma_n\). In particular,
\[ A_1 := Z_n^{(m)}(1, \theta_k) \geq T_{n-1}^{(m)}(1) \frac{\sigma_n - \theta_k}{\sigma_n} + T_{n}^{(m)}(1) \frac{\theta_k}{\sigma_n}, \]
\[ = \frac{T_{n-1}^{(m)}(1)}{\sigma_n} \left( (\sigma_n - \theta_k) + \frac{T_{n}^{(m)}(1)}{T_{n-1}^{(m)}(1)} \theta_k \right). \] (6.6)

3) Combining estimates (6.4), (6.5) and (6.6), and using \(M = n - m\), we obtain
\[ \theta_k \geq \frac{n-m}{2} \frac{\|T_{n-1}^{(m-1)}\|}{\sigma_n} \left( (\sigma_n - \theta_k) + \frac{T_{n}^{(m)}(1)}{T_{n-1}^{(m)}(1)} \theta_k \right). \]

From the relations
\[ \frac{\|T_{n-1}^{(m-1)}\|}{\sigma_n} := \frac{\|T_{n-1}^{(m-1)}\|}{\|T_{n}^{(m)}\|} = \frac{1}{2n}, \quad \frac{\|T_{n}^{(m)}\|}{T_{n-1}^{(m)}(1)} = \frac{n+1}{n-m} > \frac{n+m}{n-m}, \]
it follows further that
\[ \theta_k > \frac{n-m}{4n} \left( \sigma_n - \theta_k + \frac{n+m}{n-m} \theta_k \right) = \frac{n-m}{4n} \left( \sigma_n + \frac{2m}{n-m} \theta_k \right). \]

So, \((1 - \frac{m}{2n})\theta_k \geq \frac{n-m}{4n} \sigma_n\), and finally, since \(m = k+1\),
\[ \theta_k > \frac{n-m}{2(2n-m)} \sigma_n = \frac{n-(k+1)}{2(2n-(k+1))} \sigma_n =: \eta_k \sigma_n, \]
and that proves (6.2). \[\Box\]
Remark 6.2. For small \( n < 20 \), numerical calculations of the Zolotarev polynomials \( Z_n(x, \theta) \) such that \( Z_n^{(k+1)}(1, \theta) = 0 \) show that
\[
\theta = \eta_k \sigma_n, \quad \frac{n - (k + 1)}{n + (k + 2)} < \eta_k < \frac{n - (k + 1)}{n + (k + 1)},
\]
and that means that our constant (6.2) have the right order with respect to \( k \), and differ from the exact constant \( \eta_k \) by the factor 4 at most.

7. Schur-type inequality and an upper estimate for \( m_k^*(x_0, \sigma) \) for \( x_0 \in [\omega, 1] \). The following theorem (which appeared as Theorem 1.4) is an extension of Shur’s result (4.1) to the derivatives of any order \( 1 \leq k \leq n - 2 \).

Theorem 7.1. We have
\[
\sup_{\|p\| \leq 1, p^{(k+1)}(x_0) = 0} |p^{(k)}(x_0)| \leq \lambda_k T_n^{(k)}(1), \quad 1 \leq k \leq n - 2,
\]
where
\[
\lambda_k := \frac{1}{k + 1} \frac{n - 1}{n - 1 + k}.
\]
Proof. We proved in (4.4) that, with \( \mu_k(x_0) \) being the left-hand side of (7.1), we have
\[
\max_{x_0 \in [-1, 1]} \mu_k(x_0) = \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\},
\]
and we found in (5.1) that \( |T_n^{(k)}(\omega_k)| \leq \frac{1}{2k+1} T_n^{(k)}(1) \), whereas \( |Z_n^{(k)}(1, \theta_k)| \leq \lambda_k T_n^{(k)}(1) \) by (5.6). Since \( \lambda_k \geq \frac{1}{2k+1} \), the second estimate is maximal of two. \( \square \)

Proposition 7.2. We have
\[
\max_{x_0 \in [\omega, 1]} m_k^*(x_0, \sigma) \leq A_{n,k}(\sigma) := \lambda_k T_n^{(k)}(1) \max \left\{ 1, \frac{1}{\eta_k \sigma_n} \right\}^{k/n}, \quad 0 \leq \frac{\sigma}{\sigma_n} \leq 1,
\]
where
\[
\lambda_k := \frac{1}{k + 1} \frac{n - 1}{n - 1 + k}, \quad \eta_k := \frac{n - (k + 1)}{2(2n - (k + 1))}.
\]
Proof. We proved in (4.5) that
\[
\max_{x_0 \in [\omega, 1]} m_k^*(x_0, \sigma) \leq \max \left\{ 1, \frac{\sigma}{|\eta_k|} \right\}^{k/n} \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}.
\]
For the second maximum, as we just showed above, we have the upper bound \( \lambda_k T_n^{(k)}(1) \), and we proved in (6.1) that \( |\eta_k| \geq \sigma_n \eta_n \), hence the statement. \( \square \)

8. Upper estimates for \( m_k(x, \sigma) \) for \( x \in [0, \omega] \) and \( 0 \leq \sigma \leq \sigma_n \). For \( f \in W_\infty^k(\sigma) \), let \( \ell_\delta \in \mathcal{P}_{n-1} \) be the Lagrange polynomial of degree \( n - 1 \) that interpolates \( f \) at the mesh \( \delta = (t_i)^{n-1} \) where \( t_i = \cos \frac{\pi i}{n-1} \) are the points of local extrema of the Chebyshev polynomial \( T_{n-1} \) on the interval \([−1, 1] \), i.e.
\[
\ell_\delta(t_i) = f(t_i), \quad (t_i^2 - 1)T_{n-1}'(t_i) = 0.
\]
From the identity \( f^{(k)}(x) = \ell_\delta^{(k)}(x) + (f^{(k)}(x) - \ell_\delta^{(k)}(x)) \), it follows that
\[
|f^{(k)}(x)| \leq D_k(x) \|f\|_\delta + \Omega_k(x) \|f^{(n)}\|,
\]
Proof. The values of local maxima of $|T_{n-1}^{(m)}|$ increase towards the end-points, and with $\bar{\omega}_m := \omega_{n-1,m}$ being the rightmost zero of $|T_{n-1}^{(m+1)}|$, we have

$$\max_{x \in [0, \bar{\omega}_m]} |T_{n-1}^{(m)}(x)| \leq |T_{n-1}^{(m)}(\bar{\omega}_m)| \leq \frac{1}{2m+1} |T_{n-1}^{(m)}(1)|.$$
where we used Lemma 5.1. In particular,
\[ \max_{x \in [0, \omega_{k}]} |T_{n-1}^{(k)}(x)| \leq \frac{1}{2k+1} T_{n-1}^{(k)}(1), \]
\[ \max_{x \in [0, \omega_{k-1}]} |T_{n-1}^{(k-1)}(x)| \leq \frac{1}{2k-1} T_{n-1}^{(k-1)}(1), \]
so (8.6) is proved, but (8.5) is established for \( x \in [0, \omega_{k}] \) so far. But on the interval \([\omega_{k}, \omega_{k-1}]\) the value \( |T_{n-1}^{(k)}(x)| \) decreases monotonically from the rightmost local maximum \( |T_{n-1}^{(k)}(\omega_{k})| \) to the rightmost zero \( T_{n-1}^{(k)}(\omega_{k-1}) = 0 \), hence the inequality (8.5) for such \( x \) as well.

Next, we combine Lemmas 8.1 and 8.2 with the estimates of \( T_{n-1}^{(m)} \) in Lemma 8.3, and that allows us to estimate the right-hand side in (8.2). We note that this method was used earlier by Eriksson [3], but his choice of interpolation points was different from ours.

**Lemma 8.4.** We have
\[ \max_{x \in [0, \omega]} D_{k}(x) \leq \frac{3}{2k+1} T_{n-1}^{(k)}(1). \]

**Proof.** By Lemma 8.1, we have the estimate
\[ D_{k}(x) \leq \max\{U(x), V(x)\}, \]
where \( U(x) := |T_{n-1}^{(k)}(x)| \) and
\[
V(x) := \left\lfloor \frac{1}{k} (x^{2} - 1) T_{n-1}^{(k+1)}(x) + xT_{n-1}^{(k)}(x) \right\rfloor \\
\leq \frac{k - 1}{k} |T_{n-1}^{(k)}(x)| + \frac{(n - 1)^{2} - (k - 1)^{2}}{k} |T_{n-1}^{(k-1)}(x)|,
\]
where in the second line we used the differential equation for \( T_{n-1} \). Then, according to Lemma 8.3, we have
\[
\max_{x \in [0, \omega]} U(x) \leq \frac{1}{2k+1} T_{n-1}^{(k)}(1), \\
\max_{x \in [0, \omega]} V(x) \leq \frac{k - 1}{k} \frac{1}{2k+1} T_{n-1}^{(k)}(1) + \frac{(n - 1)^{2} - (k - 1)^{2}}{k} \frac{1}{2k-1} T_{n-1}^{(k-1)}(1) \\
= \left( \frac{k - 1}{k} \frac{1}{2k+1} + \frac{1}{k} \right) T_{n-1}^{(k)}(1) \\
= \frac{3}{2k+1} T_{n-1}^{(k)}(1).
\]

**Lemma 8.5.** We have
\[ \max_{|x| \leq \bar{\omega}} \Omega_{k}(x) \leq \frac{2}{2k+1} \frac{k(k+1)}{n+k} \frac{1}{\sigma_{n}} T_{n}^{(k)}(1). \]

**Proof.** 1) We want to apply Lemma 8.2 with \( \alpha = \bar{\omega} \), so we show first that such \( \alpha \) satisfies lemma’s condition regarding zeros of polynomials (8.4). It is clear by inspection that
- zeros of \( P_{1}(x) = (x + 1)T_{n-1}^{\prime}(x) \) precede those of \( Q(x) := T_{n-1}(x) \),
- zeros of \( Q(x) := T_{n-1}(x) \) precede those of \( P_{2}(x) = (x - 1)T_{n-1}^{\prime}(x) \),
so, by the Markov’s interlacing property, the zeros of the polynomials $P_1^{(k)}$, $P_2^{(k)}$ and $Q^{(k)}$ interlace in the same way, i.e.,

- zeros of $P_1^{(k)}$ precede those of $Q^{(k)}$,
- zeros of $Q^{(k)}$ precede those of $P_2^{(k)}$.

But $\hat{\omega}$ is the rightmost zero of the polynomial $Q^{(k)}(x) := T^{(k)}_{n-1}$, hence, it lies between the rightmost zeros of polynomials $P_1^{(k)}$ and $P_2^{(k)}$ from (8.4), and we may apply Lemma 8.2 to obtain

$$\max_{|x| \leq \hat{\omega}} \Omega_k(x) \leq \frac{1}{|x| \leq \hat{\omega}} \|v^{(k)}(x)\|,$$

where $v(x) = \frac{1}{2^{n-k}} \frac{1}{n-k} (x^2 - 1) T'_{n-1}(x)$. If we set

$$q(x) := (x^2 - 1) T'_{n-1}(x),$$

we may rewrite the latter inequality as

$$\max_{|x| \leq \hat{\omega}} \Omega_k(x) \leq \frac{1}{2^{n-k-1}} \frac{1}{n-k} \max_{|x| \leq \hat{\omega}} |q^{(k)}(x)| = \frac{2}{\sigma_n n - 1} \max_{|x| \leq \hat{\omega}} |q^{(k)}(x)|. \quad (8.7)$$

2) Now we evaluate the maximum on the right-hand side of (8.7). We have $q'(x) = (n-1)T'_{n-1}(x) + xT''_{n-1}(x)$, so differentiating further and using Lemma 8.3, we obtain

$$|x| \leq \hat{\omega} \Rightarrow |q^{(k)}(x)| \leq \frac{(n-1)^2 + (k-1)}{2k-1} |A^{(k-1)}_{n-1}(x)| + |xT''_{n-1}(x)| \leq \frac{(n-1)^2 + (k-1)}{2k+1} T'_{n-1}(1)$$

$$= \left( \frac{(n-1)^2 + (k-1)}{(n-1)^2 - (k-1)^2} + \frac{1}{2k+1} \right) T'_{n-1}(1) = \frac{c_{n,k}}{2k+1} T_{n-1}(1).$$

Here, since $T'_{n-1}(1) = \frac{n-k}{n-1+k} T_{n-1}(1)$, we have

$$c_{n,k} := \frac{2(k+1)(n-1)^2 + (k+2)(k-1)}{n-k} \frac{n-k}{n-1+k} = \frac{2(k+1)(n-1)^2 + (k+2)(k-1)}{(n-1)^2 - (k-1)^2} \frac{n-1}{n-1+k} \frac{1}{n-1+k} \leq 2(k+1)(n-1) \frac{1}{n-1+k} \frac{n-1}{n} \leq 2(k+1)(n-1) \frac{1}{n+k}.$$

Hence,

$$\max_{|x| \leq \hat{\omega}} |q^{(k)}(x)| \leq \frac{2(k+1)}{2k+1} \frac{n}{n+k} T_{n-1}(1),$$

and, from (8.7), we conclude

$$\max_{|x| \leq \hat{\omega}} \Omega_k(x) \leq \frac{2(k+1)}{2k+1} \frac{1}{n+k} \frac{1}{\sigma_n} T'_{n-1}(1).$$


**Proposition 8.6.** We have

$$\max_{|x| \leq \hat{\omega}_{k-1}} m_k(x, \sigma) \leq A_{n,k}(\sigma) := \frac{3}{2k+1} T_{n-1}(1) + \frac{2}{2k+1} \frac{2(k+1)}{n+k} T_{n-1}(1) \frac{\sigma}{\sigma_n}. \quad (8.8)$$
Proof. From (8.2) with \( \alpha = \tilde{\omega} \), by Lemmas 8.4-8.5, we obtain
\[
\max_{|x| \leq \tilde{\omega}} m_k(x, \sigma) \leq \max_{|x| \leq \tilde{\omega}} D_k(x) + \max_{|x| \leq \tilde{\omega}} \Omega_k(x) \cdot \sigma \leq A_{n,k}(\sigma).
\]

We need, however, the estimate (8.8) to be valid for \( x \in [0, \omega_k] \) rather than for \( x \in [0, \tilde{\omega}_{k-1}] \), but we get it in a trivial way from the following statement.

**Lemma 8.7.** We have
\[
\omega_k := \omega_{n,k} \leq \omega_{n-1,k-1} =: \tilde{\omega}_{k-1},
\]
i.e., the rightmost zero of \( T_{n-1}^{(k+1)} \) precedes the rightmost zero of \( T_{n-1}^{(k)} \).

*Proof.* Firstly, we note that \( T_n^\prime \) and \( T_{n-1} \) are both either odd or even, and that positive zeros of \( T_n^\prime(x) \) precede those of \( T_{n-1} \),

because
\[
1 \leq i \leq \frac{n}{2} \Rightarrow \frac{\pi i}{n} \geq \frac{\pi (2i - 1)}{2(n - 1)} \Rightarrow \cos \frac{\pi i}{n} \leq \cos \frac{\pi (2i - 1)}{2(n - 1)}.
\]

In Lemma 8.9 below, we show that, similarly to the Markov’s interlacing property, the order (8.10) for positive zeros of two symmetric polynomials implies the same order for positive zeros of their derivatives of any order \( k \), i.e.,

positive zeros of \( T_n^{(k+1)} \) precede those of \( T_{n-1}^{(k)} \),

in particular, \( \omega_k < \tilde{\omega}_{k-1} \). □

**Proposition 8.8.** We have
\[
\max_{x \in [0,\omega_k]} m_k(x, \sigma) \leq A_{n,k}(\sigma) := \frac{3}{2k + 1} T_{n-1}^{(k)}(1) + \frac{2}{2k + 1} \frac{2(k+1)}{n+k} T_n^{(k)}(1) \frac{\sigma}{\sigma_n},
\]
\[
\max_{x \in [0,\omega_k]} m_k(x, \sigma_n) \leq A_{n,k}(\sigma_n) \leq \frac{3}{2k + 1} T_n^{(k)}(1), \quad k \geq 2.
\]

*Proof.* The first inequality follows immediately from (8.8) and (8.9). Letting \( \sigma = \sigma_n \), there, we obtain
\[
A_{n,k}(\sigma_n) := \alpha_{n,k} T_n^{(k)}(1),
\]
where
\[
\alpha_{n,k} = \frac{3}{2k + 1} \frac{n - k}{n - 1 + k} + \frac{2}{2k + 1} \frac{2(k+1)}{n+k}.
\]

We then derive
\[
\alpha_{n,k} \leq \frac{3}{2k + 1} \frac{n - k}{n + k} + \frac{2}{2k + 1} \frac{2(k+1)}{n+k} = \frac{3}{2k + 1} \frac{n + \frac{1}{2}k + \frac{1}{2}}{n + k} \leq \frac{3}{2k + 1},
\]
where we used condition \( k \geq 2 \) in the last inequality. □

We finish this section with the lemma that we used in the proof of (8.11).

**Lemma 8.9.** Let polynomials \( r_1, r_2 \in \mathbb{P}_n \) be both either even or odd. If positive zeros of \( r_1 \) precede those of \( r_2 \), then, for their derivatives of any order \( k \), positive zeros of \( r_1^{(k)} \) precede those of \( r_2^{(k)} \).
Proof. By induction, it is sufficient to show that the statement is true for the first
derivatives, since $r'_1$ and $r'_2$ are also of the same symmetric form and will satisfy
preceding order.

If positive zeros $(a_i)$ of $r_1$ precede zeros $(b_i)$ of $r_2$, then we can transform $r_1$ into
$r_2$ by increasing each value $a_j$ until it reaches the value $b_j$. Thus, it is sufficient to
show that, for even or odd polynomials of the form

$$p(x) = \prod_{i=1}^{m} (x^2 - a_i^2), \quad \text{or} \quad q(x) = x \prod_{i=1}^{m} (x^2 - a_i^2), \quad a_i > 0, \quad (8.14)$$

the positive zeros of their derivatives are increasing functions with respect to each
$a_j$.

Let $t = t(a_1, \ldots, a_m) > 0$ be a positive zero of $q'$ for an odd polynomial $q$ in
(8.14). Then

$$\frac{q'(t)}{q(t)} = \frac{1}{t} + \sum_{i=1}^{m} \frac{2t}{(t^2 - a_i^2)} \equiv 0,$$

and differentiating this identity with respect to $a_j$ we obtain, with $t' = \frac{\partial \sigma}{\partial a_j}$,

$$- \frac{t'}{t^2} + \sum_{i=1}^{m} \frac{2t'(-a_i^2 - t^2)}{(t^2 - a_i^2)^2} + \frac{2t \cdot 2a_j}{(t^2 - a_j^2)^2} \equiv 0,$$

whence,

$$t' = \frac{4a_j}{(t^2 - a_j^2)^2} > 0.$$

For zeros of $p'$ of an even polynomial $p$ in (8.14), the proof is almost the same (the
term $\frac{1}{t}$ will disappear from the last expression).

9. **Upper estimates for $m_k(x, \sigma)$ for $x \in [0, \omega_k]$ and $\sigma = \sigma_n$.** For $k = 1$, the
estimate (8.13) is not of much help since we need a constant in front of $T'_n(1)$ to be
less than 1. The next lemmas will help us to deal with that case, and they will also
provide a better upper bound for $A_{n,k}(\sigma_n)$ for large $n$.

The following statement follows from the results of [9].

**Lemma 9.1** ([9]). Let $f \in W^n_\infty(\sigma_n)$, i.e., $\|f\| \leq 1$ and $\|f^{(n)}\| \leq \sigma_n := \|T_n^{(n)}\|$. Then

$$T_n^{(k+1)}(\xi_i) = 0 \Rightarrow |f^{(k)}(\xi_i)| \leq |T_n^{(k)}(\xi_i)|. \quad (9.1)$$

We will use it to prove the following estimate.

**Lemma 9.2.** We have

$$\max_{x \in [0, \omega_k]} m_k(x, \sigma_n) \leq (1 + \delta_k)^k |T_n^{(k)}(\omega_k)|, \quad (9.2)$$

where $\delta_k$ is the half of the maximal distance between two consecutive zeros of $T_n^{(k+1)}$.

Proof. Let $(\xi_i)$ be the zeros of $T_n^{(k+1)}$, and let

$$\delta := \delta_k := \frac{1}{2} \max_i |\xi_i - \xi_{i+1}|.$$

Consider $f \in W^n_\infty(\sigma_n)$, and set

$$f_\epsilon(x) := f\left(\frac{x}{1 + \epsilon}\right), \quad x \in [-1 - \epsilon, 1 + \epsilon], \quad \epsilon \in [0, \delta].$$
For any \( \epsilon \), restriction to \([-1, 1]\) of the function \( f_\epsilon \), as well as of its shift \( f_\epsilon(\cdot + \epsilon') \) with \( \epsilon' \leq \epsilon \), belongs to \( W^1_\infty(\sigma_n) \), so estimate (9.1) is applicable to \( f_\epsilon \).

Let \( \eta \) be any point in \([0, \omega_k]\), so that \( \eta \in (\xi_i, \xi_{i+1}] \) for some \( i \), with \( \xi_{i+1} > 0 \). Then we have two options.

1) If \( \eta := \eta(1+\epsilon) = \xi_{i+1} + \epsilon \), then set
\[
\eta(x) := f_{\epsilon_0}(x), \quad 0 < \epsilon_0 < \delta.
\]

2) Otherwise, we have \( \eta := \eta(1+\delta) \in (\xi_i, \xi_{i+1}] \), so \( \eta \) stays at the distance no larger than \( \delta \) from one of the end-points, say, \( \xi_{i+1} \). In that case, we set
\[
\eta(x) := f_{\epsilon_0}(x - \xi_{i+1} + \eta_0), \quad \epsilon_0 = \delta.
\]

As we mentioned above, in both cases we have \( g_\eta \in W^1_\infty(\sigma_n) \), so by (9.1),
\[
|g_\eta^{(k)}(\xi_{i+1})| = |f_{\epsilon_0}^{(k)}(\eta_0)| = (1 + \epsilon_0)^{-k}|f^{(k)}(\eta)| \leq T_\eta^{(k)}(\xi_{i+1}), \quad 0 < \epsilon < \delta.
\]
Hence
\[
|f^{(k)}(\eta)| \leq (1 + \delta)^k|T_\eta^{(k)}(\xi_{i+1})| \leq (1 + \delta)^k|T_\eta^{(k)}(\omega_k)|,
\]
and lemma is proved.

**Corollary 9.3.** We have
\[
\max_{x \in [0, \omega_k]} m_k(x, \sigma_n) \leq \hat{A}_{n,k}(\sigma_n) := \left(1 + \sin \frac{\pi (k+1)}{2n}\right)^k T_\eta^{(k)}(\omega_k). \tag{9.3}
\]

**Proof.** Since \( (\cos \frac{\pi i}{n}) \) are zeros of \( T_\eta' \), the zeros \( \xi_i \) of \( T_\eta^{(k+1)} \) are located in the intervals \( \cos \frac{\pi (i+k)}{n} < \xi_i < \cos \frac{\pi i}{n} \), and for the distance \( 2\delta_k \) between two consecutive \( \xi_i \) we have
\[
2\delta_k = \max_i |\xi_i - \xi_{i+1}| \leq \max_i \left| \cos \frac{\pi i}{n} - \cos \frac{\pi (i+k+1)}{n} \right| \leq 2\sin \frac{\pi (k+1)}{2n}.
\]
so we can take \( \delta_k = \sin \frac{\pi (k+1)}{2n} \) in (9.2). □

**Corollary 9.4.** We have
\[
\max_{x \in [0, \omega_1]} m_1(x, \sigma_n) = \alpha_{n,1} T_{\omega_1}'(1), \quad \alpha_{n,1} \leq 0.4, \quad n \geq 3. \tag{9.4}
\]

**Proof.** a) For \( n = 3 \), we have \( \omega_1 = 0 \), so \( x \in [0, \omega_1] = \{ \omega_1 \} \), and by (9.1), we have
\[
m_1(\omega_1, \sigma_n) = T_{\omega_1}'(\omega_1) = \frac{1}{3} T_{\omega_1}'(1) \quad \Rightarrow \quad \alpha_{3,1} = \frac{1}{3}.
\]

b) For \( n = 4 \), we have
\[
T_4(x) = 8x^4 - 8x^2 + 1, \quad T_4'(x) = 16(2x^3 - x), \quad T_4''(x) = 16(6x^2 - 1),
\]
so that \( \omega_1 = \delta_1 = \frac{1}{\sqrt{6}} \), and \( T_{\omega_1}'(\omega_1) = \frac{32}{3\sqrt{6}} = \frac{2}{3\sqrt{6}} T_4'(1) \), therefore, by (9.2),
\[
\alpha_{4,1} \leq \left(1 + \frac{1}{\sqrt{6}}\right) \frac{2}{3\sqrt{6}} = 0.38.
\]

c) For \( n \geq 5 \), by (5.1) we have \( T_n'(\omega_1) \leq \frac{1}{4} T_n'(1) \), and \( \sin \frac{\pi}{n} \leq \sin \frac{\pi}{5} \), hence, by (9.3),
\[
\alpha_{n,1} \leq \left(1 + \sin \frac{\pi}{5}\right)^{1/4} = 0.39, \quad n \geq 5.
\]
□
10. **Proof of Theorem 1.3: the case** $0 \leq \sigma \leq \sigma_n$, $1 \leq k \leq n - 2$. We need to prove that
\[
\max_{x \in [-1,1]} m_k(x, \sigma) = m_k(1, \sigma), \quad 1 \leq k \leq n - 2, \tag{10.1}
\]
and according to Claim 2.1 and (2.3)-(2.4) this would follow from two inequalities
\[
\max_{x_0 \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq m_k(1, \sigma), \quad 1 \leq k \leq n - 2, \tag{10.2}
\]
\[
\max_{x_0 \in [0, \omega_k]} m_k(x, \sigma) \leq m_k(1, \sigma), \quad 1 \leq k \leq n - 2, \tag{10.3}
\]
so we start to deal with each of them.

**Theorem 10.1.** We have
\[
\max_{x_0 \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n, \quad 1 \leq k \leq n - 2.
\]

**Proof.** By Proposition 7.2, we have the upper estimate
\[
\sup_{x \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq A^*_{n,k}(\sigma) := \lambda_k T^{(k)}_n(1) \max \left\{ 1, \frac{1}{\eta_k \sigma_n} \right\}^{k/n}, \quad 0 \leq \frac{\sigma}{\sigma_n} \leq 1, \tag{10.4}
\]
where
\[
\lambda_k := \frac{1}{k+1} \frac{n-1}{n-1+k}, \quad \eta_k := \frac{n-(k+1)}{2(2n-(k+1))}. \tag{10.5}
\]
By Proposition 2.2, we have the lower estimate
\[
m_k(1, \sigma) \geq B_{n,k}(\sigma) := \left(1 - \frac{\sigma}{\sigma_n}\right) T^{(k)}_{n-1}(1) + \frac{\sigma}{\sigma_n} T^{(k)}_n(1),
\]
\[
= \left(\rho_k \left(1 - \frac{\sigma}{\sigma_n}\right) + \frac{\sigma}{\sigma_n}\right) T^{(k)}_n(1), \quad 0 \leq \frac{\sigma}{\sigma_n} \leq 1, \tag{10.6}
\]
where
\[
\rho_k := \frac{T^{(k)}_{n-1}(1)}{T^{(k)}_n(1)} = \frac{n-1}{n} \frac{n-k}{n-1+k}. \tag{10.7}
\]
So, let us show that
\[
A^*_{n,k}(\sigma) \leq B_{n,k}(\sigma). \tag{10.8}
\]

1) **The case** $0 < \frac{\sigma}{\sigma_n} \leq \eta_k$. By (10.4) and (10.6), we have
\[
A_{n,k}(\sigma) \leq \lambda_k T^{(k)}_n(1), \quad B_{n,k}(\sigma) \geq B_{n,k}(0) = \rho_k T^{(k)}_n(1).
\]
We need to show that $\lambda \leq \rho$:
\[
\lambda \leq \rho \iff \frac{1}{k+1} \frac{n-1}{n-1+k} \leq \frac{n-1}{n} \frac{n-k}{n-1+k} \iff \frac{1}{k+1} \leq \frac{n-k}{n}, \tag{10.9}
\]
and the latter is valid for $k \leq n - 1$. So $A^*_{n,k}(\sigma) \leq B_{n,k}(\sigma)$ as required.

2) **The case** $\eta_k \leq \frac{\sigma}{\sigma_n} \leq 1$. Set $t = \frac{\sigma}{\sigma_n}$. Then, by (10.4)-(10.7), we have
\[
A^*_{n,k}(\sigma) \leq \lambda \left(\frac{t}{\eta}\right)^{k/n} T^{(k)}_n(1), \quad B_{n,k}(\sigma) \geq (\rho(1-t) + t) T^{(k)}_n(1),
\]
so we need to prove that
\[
f(t) := \lambda \left(\frac{t}{\eta}\right)^{k/n} \leq \rho (1-t) + t =: \ell_2(t), \quad t \in [\eta, 1].
\]
The function $f$ is concave, therefore it is bounded from above by its tangent $\ell_1$ at $t = 2\eta$, i.e.,

$$f(t) \leq \ell_1(t) := \lambda 2^{k/n} \left( 1 + \frac{k t - 2\eta}{n} \right).$$

So, we are done, once we prove that

$$\ell_1(t) \leq \ell_2(t) \quad t \in [0, 1].$$

Both functions are straight lines, so we need to check this inequality only at the end-points. We will use the trivial inequality $2k/n \leq 1 + k/n$.

1) At $t = 0$, we have

$$\ell_1(0) = \lambda 2^{k/n} \left( 1 - \frac{k}{n} \right) \leq \lambda, \quad \ell_2(0) = \rho.$$

The inequality $\lambda \leq \rho$ has been already established in (10.9), hence $\ell_1(0) \leq \ell_2(0)$.

2) At $t = 1$, we have $\ell_2(1) = 1$, while for $\ell_1(1)$, substituting the values of $\lambda$ and $\eta$ from (10.5), we obtain

$$\ell_1(1) = \lambda 2^{k/n} \left( 1 + \frac{k - 2\eta}{n} \right) = \frac{1}{k + 1} \frac{n - 1}{n - 1 + k} 2^{k/n} \left( 1 + \frac{k}{n (n - (k + 1))} \right).$$

Expression in the parentheses is less than $1 + k$, so

$$\ell_1(1) \leq \frac{n - 1}{n - 1 + k} \leq \frac{n + k}{n} \frac{n - 1}{n - 1 + k} < 1 = \ell_2(1),$$

and that completes the proof of (10.8), hence (10.2).

**Theorem 10.2.** We have

$$\max_{x \in [0, \omega_k]} m_k(x, \sigma) \leq m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n, \quad 1 \leq k \leq n - 2. \quad (10.10)$$

**Proof.** 1) The case $k = 1$. We use the upper estimate (9.4)

$$\max_{x \in [0, \omega_1]} m_1(x, \sigma) \leq \max_{x \in [0, \omega_1]} m_1(x, \sigma_n) \leq \frac{4}{10} T_1'(1), \quad n \geq 3,$$

and the lower estimate

$$m_1(1, \sigma) = m_1(1, \sigma_0) = T_{n-1}'(1) = \frac{(n-1)^2}{n^2} T_n'(1) \geq \frac{4}{9} T_n'(1), \quad n \geq 3,$$

hence (10.10) is valid for $k = 1$.

2) The case $2 \leq k \leq n - 2$. We use the upper estimate (8.8) with $t = \sigma/\sigma_n$

$$\max_{x \in [0, \omega_k]} m_k(x, \sigma) \leq A_{n,k}(\sigma) := \frac{3}{2k + 1} T_{n-1}'(1) + \frac{2}{2k + 1} \frac{2(k + 1)}{n + k} T_n'(1) =: \ell_1(t),$$

where, by (8.13),

$$\ell_1(1) = A_{n,k}(\sigma_n) \leq \frac{3}{2k + 1} T_n'(1), \quad k \geq 2,$$

and we use the same lower bound (10.6) as before

$$m_k(1, \sigma) \geq B_{n,k}(\sigma) := (1 - t) T_{n-1}'(1) + tT_n'(1) =: \ell_2(t).$$
To prove that \( \ell_1(t) \leq \ell_2(t) \) it is sufficient to compare their values at the end-points:

\[
\ell_1(0) = \frac{3}{2k+1} T_n^{(k)}(1) \leq T_n^{(k)}(1) = \ell_2(0),
\]

\[
\ell_1(1) \leq \frac{3}{2k+1} T_n^{(k)}(1) \leq T_n^{(k)}(1) = \ell_2(1), \quad k \geq 2,
\]

and that shows that (10.10) is valid for \( 2 \leq k \leq n - 2 \).

Therefore, both inequalities (10.2)-(10.3) are valid and that proves (10.1), hence Theorem 1.3 for \( k \leq n - 2 \).

11. **Proof of Theorem 1.3:** The case \( 0 \leq \sigma \leq \sigma_n, \ k = n - 1 \). Here we cover the case \( k = n - 1 \) for \( 0 \leq \sigma \leq \sigma_n \).

**Theorem 11.1.** We have

\[
\max_{x \in [0,1]} m_{n-1}(x, \sigma) = m_{n-1}(1, \sigma) = Z_n^{(n-1)}(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n. \tag{11.1}
\]

**Proof.** For \( f \in W_n^\infty(\sigma) \), let \( \ell_\sigma \in \mathcal{P}_{n-1} \) be the Lagrange polynomial of degree \( n - 1 \) that interpolates \( f \) at the points \( \tau_i := \tau_i(\sigma) \) of the local extrema of \( Z_n(\cdot, \sigma) \) on the interval \([-1, 1] \), i.e.,

\[
\ell_\sigma(\tau_i) = f(\tau_i), \quad -1 \leq \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n = 1.
\]

Then, from \( f^{(n-1)}(x) = \ell_\sigma^{(n-1)}(x) + (f^{(n-1)}(x) - \ell_\sigma^{(n-1)}(x)) \), it follows that

\[
|f^{(n-1)}(x)| \leq D_{n-1}(x, \sigma) \|f\| + \Omega_n(x, \sigma) \|f^{(n)}\|,
\]

where \( \|f\| = \max_{x \in [0,1]} |f(\tau_i)| \) and

\[
D_{n-1}(x, \sigma) := \max_{\|p_{n-1}\| \leq 1} |p_{n-1}^{(n-1)}(x)|,
\]

\[
\Omega_{n-1}(x, \sigma) := \max_{\|f^{(n)}\| \leq 1} |f^{(n-1)}(x) - \ell_\sigma^{(n-1)}(x)|.
\]

Therefore,

\[
\max_{x \in [0,1]} m_{n-1}(x, \sigma) \leq \max_{x \in [0,1]} D_{n-1}(x, \sigma) + \max_{x \in [0,1]} \Omega_{n-1}(x, \sigma) \cdot \sigma. \tag{11.2}
\]

1) It is known that the extreme value \( D_{n-1}(x, \sigma) \) (which is the same for all \( x \), since \( p_{n-1}^{(n-1)} \equiv \text{const} \)) is attained by the polynomial \( p_\sigma \in \mathcal{P}_{n-1} \) such that

\[
p_\sigma(\tau_i) = (-1)^{n-i}, \quad i = 1, \ldots, n. \tag{11.3}
\]

It is easy to see that, with

\[
v_\sigma(x) := \prod (x - \tau_i),
\]

we have

\[
p_\sigma(x) = Z_n(x, \sigma) - \frac{\sigma}{n!} v_\sigma(x).
\]

Indeed, (11.3) is clearly fulfilled for such \( p_\sigma \), and \( p_\sigma \) is of degree \( n - 1 \) because the leading coefficients of both polynomials on the right-hand side are equal to \( \sigma/n! \). Therefore,

\[
D_{n-1}(x, \sigma) = p_\sigma^{(n-1)}(1) = Z_n^{(n-1)}(1, \sigma) - \frac{\sigma}{n!} v_\sigma^{(n-1)}(1) > 0. \tag{11.4}
\]

2) For \( \Omega_{n-1}(x, \sigma) \), we show in Lemma 11.2 below that

\[
\max_{x \in [0,1]} \Omega_{n-1}(x, \sigma) = \Omega_{n-1}(1, \sigma) = \frac{1}{n!} v_\sigma^{(n-1)}(1). \tag{11.5}
\]
Thus, from (11.2) and (11.4)-(11.5), we obtain
\[
\max_{x \in [0,1]} m_{n-1}(x, \sigma) \leq \left| Z^{(n-1)}_n(1, \sigma) - \frac{\sigma}{n!} v^{(n-1)}(1) \right| + \left| \frac{\sigma}{n!} v^{(n-1)}_\sigma(1) \right| = Z^{(n-1)}_n(1, \sigma),
\]
and theorem is proved.

**Lemma 11.2.** We have
\[
\max_{x \in [0,1]} \Omega_{n-1}(x, \sigma) = \Omega_{n-1}(1, \sigma) = \frac{1}{n!} v^{(n-1)}(1).
\] (11.6)

**Proof.** For \( \Omega_{n-1}(x, \sigma) \), Floater [4, Sect. 5] showed that it has a convex majorant \( \Omega^*_n(x, \sigma) \),
\[
\Omega_{n-1}(x, \sigma) \leq \Omega^*_n(x, \sigma) := \frac{1}{n} \sum_{i=0}^{n-1} |x - \tau_i(\sigma)|,
\]
hence
\[
\max_{x \in [0,1]} \Omega_{n-1}(x, \sigma) = \max\{\Omega^*_n(0, \sigma), \Omega^*_n(1, \sigma)\}.
\]
We note that
\[
\Omega^*_n(1, \sigma) = 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma) = \frac{1}{n!} v^{(n-1)}(1) = \Omega_{n-1}(1, \sigma),
\]
so it remains to prove that \( \Omega^*_n(0, \sigma) \leq \Omega_{n-1}(1, \sigma) \), i.e., that
\[
c_1(\sigma) := \frac{1}{n} \sum_{i=1}^{n} |\tau_i(\sigma)| \leq 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma) =: c_2(\sigma).
\] (11.7)

For large \( n \), this inequality is self-evident because \( n \) equioscillation points \( \tau_i(\sigma) \) of \( Z_n(\cdot, \sigma) \) are spread sufficiently uniformly in the interval \([-1, 1]\), therefore \( c_1(\sigma) < 1 \) while \( c_2(\sigma) \to 1 \) as \( n \to \infty \). But we need it for small \( n \) as well.

We will use the monotonicity property of \( \tau_i(\sigma) \) as functions of \( \sigma \) mentioned in Sect. 3, namely
\[
\tau_i(\sigma_0) \leq \tau_i(\sigma) \leq \tau_i(\sigma_n), \quad 0 = \sigma_0 \leq \sigma \leq \sigma_n.
\] (11.8)

Here, \( \tau_i(\sigma_0) \) are zeros of \((x^2 - 1)T'_{n-1}(x)\) and \( \tau_i(\sigma_n) \) are zeros of \((x - 1)T'_{n}(x)\), therefore,
\[
\cos \frac{\pi(n-i)}{n-1} \leq \tau_i(\sigma) \leq \cos \frac{\pi(n-i)}{n}, \quad i = 1, \ldots, n-1, \quad \tau_n(\sigma) = 1.
\]

It follows that
\[
c_2(\sigma) = 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma) \geq 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma_n) = 1 - \frac{1}{n}.
\] (11.9)

On the other hand, with \( m = \lfloor \frac{n}{2} \rfloor \), we have
\[
nc_1(\sigma) = \sum_{i=1}^{m} |\tau_i(\sigma)| \leq \sum_{i=1}^{m} (-\tau_i(\sigma_0)) + \sum_{i=m+1}^{n} \tau_i(\sigma_n)
\]
\[
= \sum_{i=0}^{m-1} \cos \frac{\pi i}{n-1} + \sum_{i=0}^{m-1} \cos \frac{\pi i}{n} \leq 1 + \frac{1}{\sin \frac{\pi}{2m}}.
\] (11.10)

Here, for \( x \in \left\{ \frac{\pi}{n}, \frac{\pi}{n-1} \right\} \) we used the relations
\[
\sum_{i=0}^{m-1} \cos ix = 1 + \left( 1 + \sum_{i=1}^{m-1} \cos ix \right) = 1 + \frac{\sin(m - \frac{1}{2})x}{2 \sin \frac{1}{2}x} < 1 + \frac{1}{2 \sin \frac{\pi}{2m}}.
\]
So, from (11.9)-(11.10),
\[ c_1(\sigma) < \frac{1}{n} + \frac{1}{n \sin \frac{\pi}{2n}}, \quad c_2(\sigma) \geq 1 - \frac{1}{n}. \]

(a) For \( n \geq 6 \), this gives
\[ c_1(\sigma) \leq \frac{1}{n} + \frac{1}{n \sin \frac{\pi}{2n}} \leq \frac{1}{6} + \frac{1}{6 \sin \frac{\pi}{12}} = 0.81 < \frac{5}{6} \leq 1 - \frac{1}{n} \leq c_2(\sigma). \]

(b) For \( n = 5 \), computing directly the sums in (11.10), we obtain
\[ c_1(\sigma) \leq \frac{1}{5} \left( 1 + \cos \frac{\pi}{4} + \cos \frac{2\pi}{4} + 1 + \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} \right) = 0.76 < \frac{4}{5} \leq c_2(\sigma). \]

(c) For \( n = 2, 3, 4 \), we cannot obtain the inequality \( c_1(\sigma) \leq c_2(\sigma) \) using only the upper and lower bounds (11.8). But in these cases, the equioscillation points \( \tau(\sigma) \) are known explicitly for any \( \sigma \in [\sigma_0, \sigma_1] \). If we set \( \tilde{\sigma} := \sigma_n \cos^{2n} \frac{\pi}{2n} \), then for \( \sigma \in [\tilde{\sigma}_n, \sigma_n] \) (and for any \( n \in \mathbb{N} \)) we have
\[ \left\{ \tau_i = (1 + t) \cos \left( \frac{(n - i)\pi}{n} - t \right) \right\}_{i=1}^n, \quad t \in \left[ 0, \tan^2 \frac{\pi}{2n} \right], \]
and for \( \sigma \in [0, \tilde{\sigma}_n] \), we have
\[ n = 2, \quad \{ \tau_1 = -1, \tau_2 = 1 \} \]
\[ n = 3, \quad \{ \tau_1 = -1, \tau_2 = t, \tau_3 = 1 \}, \quad t \in \left[ 0, \frac{1}{3} \right] \]
\[ n = 4, \quad \{ \tau_1 = -1, \tau_2 = \frac{\tau + t - (1 - t^2)}{2}, \tau_3 = \frac{\tau + t + (1 - t^2)}{2}, \tau_4 = 1 \}, \quad t \in \left[ 0, \sqrt{2} - 1 \right]. \]
It is straightforward to show that, for such \( \tau_i \), inequality (11.7) is true.

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**REFERENCES**


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