Mean-variance portfolio selection under a constant elasticity of variance model

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Abstract

This paper discusses a mean-variance portfolio selection problem under a constant elasticity of variance model. A backward stochastic Riccati equation is first considered. Then we relate the solution of the associated stochastic control problem to that of the backward stochastic Riccati equation. Finally, explicit expressions of the optimal portfolio strategy, the value function and the efficient frontier of the mean-variance problem are expressed in terms of the solution of the backward stochastic Riccati equation.

Keywords: Mean-variance portfolio selection; Constant elasticity of variance model; Backward stochastic Riccati equation; Efficient frontier.

1. Introduction

Portfolio selection problem is an important issue in the theory and practice of finance. The modern portfolio selection theory can be traced back to the seminal work of Markowitz [23], where a mean-variance formulation was developed in a single-period setting with the Gaussian assumption for the distributions of individual returns. Ever since then, there has been a growing interest in extending and generalizing Markowitz’s work. Using embedding techniques, Li and Ng [20] and Zhou and Li [27] solved the mean-variance portfolio selection problem analytically in a multi-period and continuous-time setting, respectively. Recently, there has been an interest in studying the mean-variance portfolio selection problem in financial models with random parameters. See, for example, Lim and Zhou [22], Ferland and Watier [14] and Chiu and Wong [7], amongst others.

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The constant elasticity of variance (CEV) model was first introduced to the financial community by Cox [10]. It may be considered a type of random-coefficient financial models. An empirical advantage of the CEV model is that it can describe the implied volatility smile observed in option prices data. In the last three decades, some works have been done in option valuation under the CEV model. See, for example, Cox and Ross [11], Beckers [2], Duyvendov and Linetsky [12] and others. Recently, stochastic control problems in insurance and finance under the CEV model have attracted some attention. There are some previous works along this direction such as Xiao et al. [25], Gao [15], Jung and Kim [19], Liang et al. [21], Zhao and Rong [26], and others. However, most of these works mainly focused on the utility maximization problems under the CEV model. It seems that portfolio selection under the CEV model in the Markowitz mean-variance paradigm has not yet been well-explored.

In this paper, we discuss a continuous-time mean-variance portfolio selection problem with two securities, namely a risk-free bond and a risky share. The price process of the risky share is governed by a CEV model. The financial market described by the CEV model is a complete market with stochastic volatility; which can be regarded as a particular case of the work with general random market parameters in Lim and Zhou [22]. We adopt the stochastic linear-quadratic control approach as in Lim and Zhou [22] and relate the solution of the mean-variance problem under the CEV model to a backward stochastic Riccati equation (BSRE). Although Lim and Zhou [22] established a general theory of mean-variance portfolio selection problem with random parameters, the solution of the problem depends on solving the BSRE, which is difficult to solve in closed form when an explicit structure of random parameters is not specified. By making use of a particular structure given by the CEV model, a closed-form solution of the BSRE corresponding to the mean-variance portfolio selection problem is derived.

Explicit expressions of the optimal portfolio strategy, the value function and the efficient frontier of the mean-variance problem are then represented in terms of the solution of the BSRE.

2. Problem formulation

Let $T$ be a finite time parameter set $[0,T]$ where $T < \infty$. As usual, a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is considered, where $\mathcal{P}$ is a real-world probability measure and the expectation with respect to $\mathcal{P}$ is denoted as $E[\cdot]$. Let $\{W(t) | t \in T\}$ be a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$. Assume that $\mathcal{F} := \{\mathcal{F}(t) | t \in T\}$ is the right continuous, $\mathcal{P}$-complete filtration generated by $\{W(t) | t \in T\}$.

For any nonnegative $\mathbb{F}$-adapted process $\{a(t) | t \in T\}$, let $\{A(t) | t \in T\}$ be an increasing continuous process defined by $A(t) := \int_0^t a^2(s)ds$, $t \in T$. Let $\eta \geq 0$ be a generic constant, which may be different from line to line. On the filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F})$, we denote by $L^2_F(\eta,a;T;\mathbb{R})$ the space of all real-valued, $\mathbb{F}$-adapted processes $\{f(t) | t \in T\}$ such that $\|f\|^2 := E[\int_0^T e^{\eta A(t)}|f(t)|^2dt] < \infty$, by $L^2_{\mathcal{F}}(\eta,a;T;\mathcal{F})$ the space of all real-valued, $\mathcal{F}$-adapted processes $\{f(t) | t \in T\}$ such that $\|f\|^2_{\mathcal{F},a} := \|a\|^2_{\mathcal{F}} = E[\int_0^T a^2(t)e^{\eta A(t)}|f(t)|^2dt] < \infty$, and by $L^2_{\mathcal{F}}(\eta,a;T;\mathcal{F})$ the space of all real-valued, $\mathcal{F}$-adapted, cadlag processes $\{f(t) | t \in T\}$ such that $\|f\|^2_{\mathcal{F},c} := E[\sup_{0 \leq t \leq T}e^{\eta A(t)}|f(t)|^2] < \infty$. Then

$$M^2_F(\eta,a;T;\mathcal{F}) := (L^2_{\mathcal{F}}(\eta,a;T;\mathcal{F}) \cap L^2_{\mathcal{F}}(\eta,a;T;\mathcal{F})) 	imes L^2_{\mathcal{F}}(\eta,a;T;\mathbb{R})$$
is a Banach space with the norm \( \| (y, z) \|_\alpha^2 := \| y \|_{\alpha, 0}^2 + \| y \|_{\alpha, c}^2 + \| z \|_{\alpha}^2 \). In addition, we denote by \( C^{1,2}(T \times \mathbb{R}^+; \mathbb{R}) \) the space of real-valued continuous functions on \( T \times \mathbb{R}^+ \) with continuous derivative in the first argument and continuous derivatives up to order 2 in the second argument, and by \( C^1(T; \mathbb{R}) \) the space of continuously differentiable functions from \( T \) to \( \mathbb{R} \).

We consider a financial market consisting of a risk-free bond and a risky share. The price process of the risk-free bond \( \{ B(t) | t \in T \} \) evolves over time as:

\[
dB(t) = r(t)B(t)dt, \quad t \in T, \quad B(0) = 1,
\]

where \( r(t) \) represents the risk-free, instantaneous interest rate at time \( t \). Assume that there exists an \( \epsilon > 0 \) such that \( r(t) \geq \epsilon \), for each \( t \in T \).

The price process of the risky share \( \{ S(t) | t \in T \} \) satisfies the following stochastic differential equation (SDE):

\[
dS(t) = S(t) \left[ \mu(t)dt + S^\beta(t) \sigma(t)dW(t) \right], \quad t \in T, \quad S(0) = s > 0.
\]

Here, \( \mu(t) \) is the appreciation rate of the share; \( \sigma(t) > 0 \) can be interpreted as the scale parameter of the share; \( \beta \) is called the elasticity parameter of the share. Then, \( S^\beta(t) \sigma(t) \) represents the instantaneous volatility of the share at time \( t \). Furthermore, we require that \( r(t), \mu(t), \sigma(t) \) are deterministic, uniformly bounded functions of time \( t \).

Throughout this paper, we only consider the case of a negative elasticity parameter (i.e. \( \beta < 0 \)) for the following two reasons: Firstly, as indicated by Heston et al. [16], there are arbitrage opportunities and asset price bubbles on both option values and share prices in the case that \( \beta > 0 \). A bubble may be characterized by a price process and, when discounted, is a local martingale under the risk-neutral measure but not a martingale (see, for example, Cox and Hobson [9]). Secondly, the instantaneous volatility increases as the share price increases in the case of a positive elasticity parameter, which is not consistent with the empirical evidence of the leverage effect, (see, for example, Christie [8]).

From Lemma 6.4.4.1 in Jeanblanc et al. [17], the zero boundary is reached almost surely for the CEV model with \( \beta < 0 \) and it is an absorbing state. Consequently, the CEV model with \( \beta < 0 \) may be used to describe default. If we assume that the share price process \( S \) is killed at the first hitting time of zero and is sent to the absorbing state, the default time is defined as \( T_0 := \inf \{ t \geq 0 | S(t) = 0 \} \). Indeed, this kind of default might be related to defaults described by the Merton structural firm value model, (see Merton [24]). The structural model is intuitively appealing since it links defaults to the firm’s capital structure. However, unlike the reduced-form approach where default times are totally inaccessible, the default time \( T_0 \) is predictable with respect to the underlying filtration generated by information about the firm’s value. That is, the default event may be predicted by observing the dynamics of the firm’s value. This counterfactual feature leads to discrepancy between the credit spreads from structural models and the market data (see Jones et al. [18]). To overcome the predictability of default under the CEV model, one may consider adding a jump-to-default part in the CEV model (2) as in Campi et al. [5] or Carr and Linetsky [6]. Under such jump-to-default CEV model, defaults could be either expected (predictable) or unexpected (totally inaccessible) depending on whether they are triggered by diffusion and jump terms, respectively. This may provide
a possible way to combine the advantages of both structural and reduced-form models for credit risk.

In what follows, we consider the situation where an economic agent invests his wealth into the financial market as described by Eqs. (1) and (2). Let \( \pi(t) \) be the amount of the agent’s wealth invested in the risky share at time \( t \). Here \( \pi(\cdot) := \{ \pi(t) | t \in T \} \) is called a portfolio strategy of the agent. Let \( X(t) := X^\pi(t) \) be the total wealth of the agent corresponding to the portfolio strategy \( \pi(\cdot) \). Suppose that the portfolio strategy is self-financed. Then the wealth process \( \{X(t) | t \in T \} \) of the agent is governed by the following SDE:

\[
\begin{aligned}
&dX(t) = [r(t)X(t) + \pi(t)(\mu(t) - r(t))] \, dt + \pi(t)\sigma(t)dW(t) \quad t \in T, \\
&X(0) = x.
\end{aligned}
\]  

**Definition 2.1.** A portfolio strategy \( \pi(\cdot) \) is said to be admissible if (1) \( \pi(\cdot) \) is \( \mathbb{F} \)-adapted; (2) \( \mathbb{E} \left[ \int_0^T \pi^2(t)S(t)dt \right] < \infty; \) (3) the SDE (3) has a unique strong solution \( X(\cdot) \) corresponding to \( \pi(\cdot) \). The set of all admissible portfolio strategies is denoted by \( \mathcal{A} \).

The agent’s objective is to find an admissible portfolio \( \pi(\cdot) \in \mathcal{A} \) to minimize the variance of terminal wealth for a given level of the expected terminal wealth. Finding such a portfolio \( \pi(\cdot) \) is referred to the mean-variance portfolio selection problem. Specifically, as in the literature, the mean-variance portfolio selection problem is formulated as follows:

**Definition 2.2.** The mean-variance portfolio selection is the following constrained stochastic optimization problem, parameterized by \( d \in \mathbb{R} \): 

\[
\begin{aligned}
&\min_{\pi(\cdot) \in \mathcal{A}} \quad J(x,s;\pi(\cdot)) = \mathbb{E}[X(T) - d]^2, \\
&\text{subject to} \quad \mathbb{E}[X(T)] = d, \\
&\quad (X(\cdot),\pi(\cdot)) \text{ satisfy (3).}
\end{aligned}
\]  

(4)

As widely adopted in the literature, we apply the Lagrangian multiplier technique to deal with the constraint \( \mathbb{E}[X(T)] = d \). Define

\[
\begin{aligned}
J(x,s;\pi(\cdot),\lambda) &:= \mathbb{E}[X(T) - d]^2 + 2\lambda\mathbb{E}[X(T) - d] \\
&= \mathbb{E}[X(T) + \lambda - d]^2 - \lambda^2, \quad \lambda \in \mathbb{R}.
\end{aligned}
\]

Thus by the well-known Lagrangian duality theorem, we know that the original mean-variance portfolio selection problem (4) is equivalent to the following max-min stochastic control problem:

\[
\begin{aligned}
&\max_{\lambda \in \mathbb{R}} \min_{\pi(\cdot) \in \mathcal{A}} \quad J(x,s;\pi(\cdot),\lambda) = \mathbb{E}[X(T) + \lambda - d]^2 - \lambda^2, \\
&\text{subject to} \quad (X(\cdot),\pi(\cdot)) \text{ satisfy (3).}
\end{aligned}
\]  

(5)

Clearly, to solve the above max-min stochastic control problem (5), we need to first consider the following quadratic loss minimization problem:

\[
\begin{aligned}
&\min_{\pi(\cdot) \in \mathcal{A}} \quad J_0(x,s;\pi(\cdot),c) = \mathbb{E}[X(T) - c]^2, \\
&\text{subject to} \quad (X(\cdot),\pi(\cdot)) \text{ satisfy (3),}
\end{aligned}
\]  

(6)
where \( c := d - \lambda \).

Note that if \( \beta = 0 \), the price process of the risky share reduces to the classical geometric Brownian motion and the mean-variance portfolio selection problem can be solved via the standard stochastic LQ framework (see Zhou and Li [27]). In this paper, we are interested in the CEV model with \( \beta < 0 \). So the instantaneous volatility process \( \{ S^2(t) \sigma(t) \mid t \in T \} \) is a stochastic process and depends on the price process of the risky share (2).

3. Explicit solution of the backward stochastic Riccati equation

We consider a linear backward stochastic differential equation (BSDE):

\[
\begin{align*}
 dq(t) &= \left\{ \frac{2r(t) - (\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)} q(t) + \frac{2(\mu(t) - r(t))}{S^{2\beta}(t)\sigma(t)} \Gamma(t) \right\} dt + \Gamma(t) dW(t), \\
 q(T) &= 1.
\end{align*}
\]

(7)

Here \( (q(\cdot), \Gamma(\cdot)) \) is the solution pair which will be determined below.

Since the coefficients in the driver of (7) are random and unbounded, the driver is not uniformly Lipschitz. The classical theory of BSDE in El Karoui et al. [13] cannot be applied in this situation. Indeed, the driver of (7) satisfies the stochastic Lipschitz condition discussed in Bender and Kohlmann [3]. The existence and uniqueness result can also be established under certain conditions. The following assumption is crucial not only for the existence and uniqueness of solutions of BSDE (7) and BSRE given below, but also for the derivation of explicit expressions of these solutions.

(H) For a sufficiently large \( \eta \geq 0 \),

\[
E \left[ \exp \left\{ \eta \int_0^T \frac{(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)} dt \right\} \right] < \infty.
\]

The following lemma results from the theory of BSDEs with stochastic Lipschitz condition established by Bender and Kohlmann [3].

Lemma 3.1. Under Assumption (H), the BSDE (7) admits a unique solution \( (q(\cdot), \Gamma(\cdot)) \in M_2^T(\eta, a; \mathbb{T}; \mathbb{R} \times \mathbb{R}) \).

Proof. Put \( \alpha_0(t) = 0, \alpha_1(t) = 2r(t) - \frac{(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)} \), and \( \alpha_2(t) = \frac{2(\mu(t) - r(t))}{S^{2\beta}(t)\sigma(t)} \). We define by

\[
a^2(t) := |\alpha_0(t)| + |\alpha_1(t)| + |\alpha_2(t)|^2
\]

\[
= \left| 2r(t) - \frac{(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)} \right| + \frac{4(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)}.
\]

It is clear that

\[
2\epsilon \leq 2|r(t)| + \frac{3(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)} \leq a^2(t) \leq 2|r(t)| + \frac{5(\mu(t) - r(t))^2}{S^{2\beta}(t)\sigma^2(t)}.
\]

(8)

Combining the boundedness of \( r(t) \), Assumption (H) and (8), it can be verified that the pair of the driver and the terminal value of Eq. (7) constitute a standard data (please refer to Definition 2 in Bender and Kohlmann [3]). Therefore, from Theorem 3 in Bender and Kohlmann [3], the linear BSDE (7) admits a unique solution \( (q(\cdot), \Gamma(\cdot)) \in M_2^T(\eta, a; \mathbb{T}; \mathbb{R} \times \mathbb{R}) \).
Remark 3.1. In Bender and Kohlmann [3], a sufficiently large constant $\zeta \geq 0$ is taken such that the contraction mapping theorem can be used to prove the uniqueness and existence of a solution to the BSDE with stochastic Lipschitz condition. In the proof of Theorem 3 in Bender and Kohlmann [3], when $\frac{12}{\zeta} + \frac{9}{\zeta} < 1$, a contraction map can be constructed. From Eq. (8), $E[\exp(\zeta A(T))] \leq CE\left[\exp\left(5\zeta \int_0^T \frac{(u(t) - r(t))^2}{\sigma^2(t)} dt\right)\right]$, where $A(T) = \int_0^T a(t)dt$ and $C$ is a positive constant independent of $\zeta$. So if Assumption (H) is satisfied with $\eta \geq 5\zeta > 5(3 + \sqrt{21})$, then $E[\exp(\zeta A(T))] < \infty$ and thus the BSDE (7) admits a unique solution.

In what follows, we shall present an explicit solution of BSDE (7). We first define a process $\{Z(t) | t \in T\}$ by putting $Z(t) = \frac{1}{\sqrt{\sigma(t)}}, t \in T$. A simple application of Itô’s differentiation rule gives:

$$dZ(t) = [\beta(2\beta + 1)\sigma^2(t) - 2\beta\mu(t)Z(t)]dt - 2\beta\sigma(t)\sqrt{Z(t)}dW(t).$$

Indeed, $\{Z(t) | t \in T\}$ is a square-root process, which will then lead to explicit solutions of BSDE (7) and BSDE below.

Lemma 3.2. The solution $(q(\cdot), \Gamma(\cdot))$ of BSDE (7) is given by

$$\left\{ \begin{align*}
q(t) &= \exp \left\{ - \int_t^T 2r(u) du \right\} V(t, Z(t)), \\
\Gamma(t) &= -2\beta\sigma(t)\sqrt{Z(t)}K(t)q(t),
\end{align*} \right. \tag{10}$$

where the function $V(\cdot, \cdot) \in C^{1,2}(T \times \mathbb{R}^+; \mathbb{R})$ satisfies

$$V(t, z) = \exp \left\{ K(t)z + M(t) \right\}. \tag{12}$$

Here $K(\cdot) \in C^1(T; \mathbb{R})$ and $M(\cdot) \in C^1(T; \mathbb{R})$ are the solutions of the following system of ODEs:

$$\left\{ \begin{align*}
\frac{dK}{dt} + 2\beta(\mu(t) - 2r(t))K + 2\beta(2\beta + 1)\sigma^2(t)K^2 + \frac{(\mu(t) - r(t))^2}{\sigma^2(t)} &= 0, & K(T) = 0, \tag{13} \\
\frac{dM}{dt} + \beta(2\beta + 1)\sigma^2(t)K &= 0, & M(T) = 0. \tag{14}
\end{align*} \right.$$  

Proof. The proof is standard. For the sake of completeness we give the proof here. Using the Feynman-Kac formula, $q(t)$ can be represented in the following expectation form

$$q(t) = e^{-\int_t^T 2r(u) du} \tilde{E} \left[ \exp \left\{ \int_t^T \frac{(\mu(u) - r(u))^2}{\sigma^2(u)} Z(u) du \right\} \right] \mathcal{F}(t), \tag{15}$$

where $\tilde{E}[]$ is the expectation under a probability measure $\tilde{P}$ equivalent to $P$ on $\mathcal{F}(T)$ defined by putting

$$\frac{d\tilde{P}}{dP} |_{\mathcal{F}(T)} = \exp \left\{ - \int_0^T \frac{2(\mu(t) - r(t))^2}{\sigma^2(t)} Z(t) dt - \int_0^T \frac{2(\mu(t) - r(t))}{\sigma(t)} \sqrt{Z(t)} dW(t) \right\}.$$
From Assumption \((H)\), the Novikov condition is satisfied. By Girsanov’s theorem,
\[
\tilde{W}(t) := W(t) + \int_{0}^{t} \frac{2(\mu(u) - r(u))}{\sigma(u)} \sqrt{Z(u)} du, \quad t \in T,
\]
is a standard Brownian motion on \((\Omega, \mathcal{F}, \tilde{\mathbb{P})}\). Then under \(\tilde{\mathbb{P}}\), \(\{Z(t) | t \in T\}\) satisfies:
\[
dZ(t) = \left[\beta(2\beta + 1)\sigma^2(t) + 2\beta(\mu(t) - 2r(t))Z(t)\right] dt - 2\beta\sigma(t) \sqrt{Z(t)} d\tilde{W}(t).
\]
Since the process \(\{Z(t) | t \in T\}\) is Markovian with respect to \(\mathbb{F}\), we have
\[
q(t) = e^{-\int_{0}^{T} 2r(u) du} V(t,Z(t)),
\]
where
\[
V(t,z) = \tilde{E}_{t,z} \left[ \exp \left\{ -\int_{t}^{T} \frac{(\mu(u) - r(u))^2}{\sigma^2(u)} Z(u) du \right\} \right],
\]
where \(\tilde{E}_{t,z}[\cdot]\) is the conditional expectation under \(\tilde{\mathbb{P}}\) given that \(Z(t) = z\).

Applying the Feynman-Kac formula gives the following partial differential equation governing \(V\):
\[
\begin{align*}
\frac{\partial V}{\partial t} &+ \beta(2\beta + 1)\sigma^2(t) + 2\beta(\mu(t) - 2r(t))z \frac{\partial V}{\partial z} + 2\beta^2\sigma^2(t)z \frac{\partial^2 V}{\partial z^2} \\
&+ \frac{(\mu(t) - r(t))^2}{\sigma^2(t)} z V = 0, \quad V(T,z) = 1.
\end{align*}
\]
We try the following exponential affine form for the solution of \(V\):
\[
V(t,z) = \exp \left\{ K(t)z + M(t) \right\}.
\]
Substituting (20) into (19) and matching the coefficients immediately give the desired results (13)-(14). Thus from (17), the first component \(q\) of the solution of BSDE (7) is given by (10) and (12). Applying Itô’s differentiation rule to \(q(t)\) gives
\[
dq(t) = \left\{ \cdots \right\} dt - 2\beta\sigma(t) \sqrt{Z(t)} K(t) q(t) d\tilde{W}(t).
\]
Comparing the diffusion term of (21) with that of (7) leads to the desired result (11). \(\square\)

Now, we define the BSRE through the linear BSDE (7). Let \(\{p(t) | t \in T\}\) be the reciprocal process of \(\{q(t) | t \in T\}\), i.e. \(p(t) := \frac{1}{q(t)}\). Applying Itô’s differentiation rule to \(p(t)\) gives the following BSRE:
\[
\begin{align*}
dp(t) &= \left\{ \begin{array}{l}
-2r(t) + \frac{(\mu(t) - r(t))^2}{S^{2\beta(t)} \sigma^2(t)} p(t) + \frac{2(\mu(t) - r(t))}{S^{\beta(t)} \sigma(t)} \Lambda(t) \\
+ \frac{\Lambda^2(t)}{p(t)} \end{array} \right\} dt + \Lambda(t) d\tilde{W}(t),
\end{align*}
\]
\(p(T) = 1\),
Since \((p(\cdot), \Lambda(\cdot))\) has a one-to-one correspondence with the solution of BSDE (7), BSRE (22) also admits a unique solution.

**Lemma 3.3.** The solution \((p(\cdot), \Lambda(\cdot))\) of BSRE (22) is given by

\[
\begin{align*}
p(t) &= \exp\left\{ \int_t^T 2r(u)du - K(t)Z(t) - M(t) \right\}, \quad \text{(24)} \\
\Lambda(t) &= 2\beta\sigma(t)\sqrt{Z(t)}K(t)p(t), \quad \text{(25)}
\end{align*}
\]

where \(K(\cdot) \in C^1(T;\mathbb{R})\) and \(M(\cdot) \in C^1(T;\mathbb{R})\) are the solutions of the ODEs (13) and (14), respectively.

**Proof.** The proof is again standard. Eqs. (24)-(25) can be immediately obtained from the relationship \(p(t) = \frac{1}{q(t)}\) and Eq. (23). \(\square\)

Although Eq. (13) is a Riccati ODE and is difficult to solve in general, we can obtain an explicit representation of the solution of (13) using Radon’s Lemma in Abou-Kandil et al. [1]. The Radon lemma has been used in the literature such as Chiu and Wong [7] to turn a matrix-valued Riccati ODE into a solvable linear ODE.

**Lemma 3.4.** Denote by \(\tau := T - t \in T\). If \(K(t)\) and \(M(t)\) satisfy (13)-(14), then the explicit solutions of \(K(\cdot)\) and \(M(\cdot)\) are given as follows:

\[
\begin{align*}
K(t) &= \frac{R_2(\tau)}{R_1(\tau)}, \quad \text{(26)} \\
M(t) &= \beta(2\beta + 1)\int_0^\tau \sigma^2(u)K(u)du, \quad \text{(27)}
\end{align*}
\]

where \(R_1(\cdot), R_2(\cdot) \in C^1(T;\mathbb{R})\) and \(R(\tau) := (R_1(\tau), R_2(\tau))^\top\) is the fundamental matrix solution of the following linear system of ODEs:

\[
\frac{dR}{d\tau} = \begin{pmatrix}
-\beta(\mu(\tau) - 2r(\tau)) & -2\beta^2\sigma^2(\tau) \\
\frac{\mu(\tau) - r(\tau)^2}{\sigma^2(\tau)} & \beta(\mu(\tau) - 2\sigma(\tau))
\end{pmatrix} R,
\]

with initial conditions \(R_1(0) = 1\) and \(R_2(0) = 0\). Particularly, if all the coefficients \(r(\cdot), \mu(\cdot), \sigma(\cdot)\) in the model dynamics are constants, i.e. \(r(\cdot) := r, \mu(\cdot) := \mu\) and \(\sigma(\cdot) := \sigma\), for each \(t \in T\), the solution of \(R(\tau)\) is given by

\[
R(\tau) = \exp\left[ \begin{pmatrix}
-\beta(\mu - 2r) & -2\beta\sigma^2 \\
\frac{(\mu - r)^2}{\sigma^2} & \beta(\mu - 2\sigma)
\end{pmatrix} \tau \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

**Proof.** The desired results (26) and (28) follow directly from Radon’s Lemma. Once the solution of \(K(t)\) is obtained, it is not difficult to derive that the solution of \(M(t)\) is given by (27). When \(r(\cdot) = r, \mu(\cdot) = \mu\) and \(\sigma(\cdot) = \sigma\) are constants, Eq. (28) becomes an autonomous linear system of ODEs, whose solution is obviously given by the matrix exponential (29). \(\square\)
Corollary 3.1. If \( r(t) = r, \mu(t) = \mu \) and \( \sigma(t) = \sigma \), the closed-form solutions of \( K(\cdot) \) and \( M(\cdot) \) are given by

\[
K(t) = \begin{cases} 
\frac{(\mu - r)^2(T - t) \sin \delta}{\cos \delta - \frac{\beta(\mu - 2r)(T - t)}{\delta} \sin \delta}, & \mu < -\sqrt{2}r \text{ or } \mu > \sqrt{2}r, \\
\frac{(\mu - r)^2}{T - t} \cos \delta + \frac{\beta(\mu - 2r)(T - t)}{\delta} \cos \delta, & \mu = -2\sqrt{r} \text{ or } \mu = 2\sqrt{r}, \\
\frac{(\mu - r)^2}{(T - t)^2} \sinh \delta, & -\sqrt{2}r < \mu < \sqrt{2}r. 
\end{cases}
\]

and

\[
M(t) = \begin{cases} 
\frac{2\beta + 1}{2\beta} \log \left| \cos \delta - \frac{(\mu - 2r)(T - t) \sin \delta}{\delta} \right|, & \mu < -\sqrt{2}r \text{ or } \mu > \sqrt{2}r, \\
\frac{2\beta + 1}{2\beta} \log \left| 1 - \frac{(\mu - 2r)(T - t)}{\delta} \right|, & \mu = -2\sqrt{r} \text{ or } \mu = 2\sqrt{r}, \\
\frac{2\beta + 1}{2\beta} \log \left| \cos \delta - \frac{(\mu - 2r)(T - t) \sin \delta}{\delta} \right|, & -\sqrt{2}r < \mu < \sqrt{2}r. 
\end{cases}
\]

where \( \Delta := 4\beta^2(-\mu^2 + 2r^2)(T - t)^2 \) and \( \delta := \frac{1}{2} \sqrt{\Delta} \).

Proof. The \((2 \times 2)\)-matrix exponential in Eq. (29) can be calculated explicitly as follows (see Bernstein and So [4] for more detailed calculations)

\[
\exp \left[ \begin{pmatrix} -\beta(\mu - 2r) & -2\beta^2 \sigma^2 \\ (\sigma^2 r^2) & \beta(\mu - 2r) \end{pmatrix} t \right] = \begin{pmatrix} \cos \delta - \frac{\beta(\mu - 2r)T}{\sigma^2 r^2} \sinh \delta & -2\beta^2 \sigma^2 \sin \delta \\ \frac{\sigma^2 r^2}{\delta} \sinh \delta & \cos \delta + \frac{\beta(\mu - 2r)T}{\sigma^2 r^2} \sin \delta \end{pmatrix}, \quad \Delta < 0,
\]

\[
= \begin{pmatrix} 1 - \beta(\mu - 2r)T & -2\beta^2 \sigma^2 T \\ \frac{\sigma^2 r^2}{\delta} \sinh \delta & 1 + \beta(\mu - 2r)T + \frac{\beta(\mu - 2r)T}{\sigma^2 r^2} \sin \delta \end{pmatrix}, \quad \Delta = 0, \quad (32)
\]

Combining (26)/(29)/(32) gives the explicit expression (30) for \( K(t) \). Furthermore, from (28),

\[
\frac{dR_1(t)}{dt} = \beta(\mu - 2r)R_1(t) + 2\beta^2 \sigma^2 R_2(t).
\]
Substituting this into (27) yields
\[ M(t) = \frac{2\beta + 1}{2\beta} \log|R_1(\tau)| - \frac{(2\beta + 1)(\mu - 2r)\tau}{2}. \]

Using (32) results in the explicit expression (31) for \( M(t) \).

4. Solutions of the quadratic loss minimization problem and mean-variance problem

The following lemma presents the optimal portfolio strategy and the value function of the quadratic loss minimization problem via the solution of BSRE (22).

Lemma 4.1. Let \( K(\cdot) \) be the solution of the Riccati equation (13). Under Assumption (H), the quadratic loss minimization problem (6) has a unique optimal feedback control

\[ \pi^*(t) = -\left( \frac{\mu(t) - r(t)}{S^{2\beta}(t)\sigma^2(t)} + \frac{2\beta K(t)}{S^{2\beta}(t)} \right) \left[ X(t) - ce^{-\int_0^t r(u) du} \right]. \]  

Furthermore, the corresponding value function is given by

\[ J_0(x; s; \pi^*(\cdot), c) = p(0) \left( x - ce^{-\int_0^T r(u) du} \right)^2, \]

where the explicit expression of \( p(0) \) is given by Eq. (24).

Proof. The techniques used in the proof are standard. Applying Itô’s differentiation rule to \( Y(t) := X(t) - ce^{-\int_0^t r(u) du} \) gives

\[ dY(t) = \left[ r(t)Y(t) + \pi(t)(\mu(t) - r(t)) \right] dt + \pi(t)S^2(t)\sigma(t)dW(t), \]

\[ Y(0) = x - ce^{-\int_0^T r(u) du}. \]

It is clear that the quadratic loss minimization problem (6) is equivalent to

\[ \min_{\pi(\cdot) \in A} J_0(y; s; \pi(\cdot)) = E[Y^2(T)], \]

subject to \((Y(\cdot), \pi(\cdot)) \) satisfy (35).

Applying Itô’s differentiation rule \( p(t)Y^2(t) \), using the localization technique and taking expectations and integrations give

\[ E[Y^2(T)] - p(0) \left( x - ce^{-\int_0^T r(u) du} \right)^2 \]

\[ = E \left[ \int_0^T p(t) \left( S^2(t)\sigma(t)\pi(t) + \frac{1}{S^2(t)\sigma(t)} \left[ (\mu(t) - r(t)) + S^2(t)\sigma(t)\frac{\Lambda(t)}{p(t)} \right] Y(t) \right)^2 dt \right]. \]

Recalling the explicit expressions of \((p(\cdot), \Lambda(\cdot)) \) given in Lemma 3.3, it follows immediately that the unique optimal feedback control and the value function are given by (33) and (34), respectively.
Based on the above analysis, we conclude this section by giving the closed-form solution of the optimal portfolio strategy, the value function and the efficient frontier of the original mean-variance portfolio selection problem (4) in the following theorem.

**Theorem 4.1.** The optimal portfolio strategy and the corresponding value function of the mean-variance portfolio selection problem (4) are given by

\[
\pi^*(t) = -\left( \frac{\mu(t) - r(t)}{S(t)\sigma(t)} + 2\bar{\beta}K(t) \right) \left( X(t) - (d - \lambda^*)e^{-\int_0^T r(u)du} \right),
\]

(37)

and

\[
J(x,s;\pi^*(\cdot),\lambda^*) = \frac{p(0)e^{-\int_0^T r(u)du}}{1 - p(0)e^{-\int_0^T r(u)du}}(d - xe^{\int_0^T r(u)du})^2,
\]

(38)

respectively, where

\[
\lambda^* = \frac{p(0)e^{-\int_0^T r(u)du}(x - de^{\int_0^T r(u)du})}{p(0)e^{-\int_0^T r(u)du} - 1}.
\]

(39)

Furthermore, the efficient frontier of the problem satisfies

\[
E[X(T)] = \sqrt{\frac{1 - p(0)e^{-\int_0^T r(u)du}}{p(0)e^{-\int_0^T r(u)du}} \sqrt{\text{Var}[X(T)]} + xe^{\int_0^T r(u)du}}.
\]

(40)

**Proof.** To solve the original mean-variance portfolio selection problem, we only need to maximize the following performance functional

\[
J(x,s;\pi^*(\cdot),\lambda) = p(0)(x - (d - \lambda)e^{-\int_0^T r(u)du})^2 - \lambda^2,
\]

(41)

over \(\lambda \in \mathbb{R}\). Note that

\[
\frac{\partial^2 J}{\partial \lambda^2}(x,s;\pi^*(\cdot),\lambda) = 2e^{\int_0^T r(u)du}p(0) - 2 < 0.
\]

Using the first order condition to \(J(x,s;\pi^*(\cdot),\lambda)\) with respect to \(\lambda\) gives the optimal value of \(\lambda\) in Eq. (39). Substituting \(\lambda^*\) into (33) and (41) yields the optimal portfolio strategy (37) and the value function (38). Consequently, the efficient frontier can be derived from (38).

\(\square\)

**Remark 4.1.** The optimal portfolio process (37) can be decomposed into two parts, where the first part resembles to that in the classical geometric Brownian motion model, while the second part is used to hedge the volatility risk in the CEV model. The form of the efficient frontier (40) is standard and has appeared in the literature on mean-variance portfolio selection. In particular, the graph of \(E[X(T)]\) against \(\text{Var}[X(T)]\) is a concave and increasing function with intercept \(xe^{\int_0^T r(u)du}\) which is the future value of an initial investment of \(x\) in the bond.
5. Conclusion

Due to some technical difficulties, we only considered the mean-variance problem with a single share. One of the potential research topics in the future is to extend the results to the case of multiple shares. Another potential research topic is the mean-variance approach to asset-liability management under the CEV model. This may be of interest to insurance companies where an uncontrollable liability is added into a portfolio for a further analysis.

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References

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