Robust Sampled-Data $H_{\infty}$ Control for Vehicle Active Suspension Systems

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Abstract—This brief investigates the problem of robust sampled-data $H_{\infty}$ control for active vehicle suspension systems. By using an input delay approach, the active vehicle suspension system with sampling measurements is transformed into a continuous-time system with a delay in the state. The transformed system contains non-differentiable time-varying state delay and polytopic parameter uncertainties. A Lyapunov functional approach is employed to establish the $H_{\infty}$ performance, and the controller design is cast into a convex optimization problem with linear matrix inequality (LMI) constraints. A quarter-car model is considered in this brief and the effectiveness of the proposed approach is illustrated by a realistic design example.

Index Terms—Active suspension system, convex polytope, $H_{\infty}$ control, parameter uncertainty, sampled-data control.

I. INTRODUCTION

VEHICLE suspensions have been a hot research topic due to their important roles in vehicle performance. Performance requirements for vehicle suspensions include ride comfort, road holding and suspension deflection. However, these requirements are often conflicting, and a compromise of the requirements must be reached. To this end, a considerable amount of research has been carried out [6], [16]. Among the proposed solutions, active suspension is an important approach to improve suspension performance and has a great potential to meet the requirements demanded by users [3], [7], [22]. The aim of active suspension control is to improve ride comfort, while keeping suspension stroke and tire deflection within an acceptable level. A number of active suspension control approaches have been proposed, based on various control techniques, such as fuzzy logic and neural network control [1], $H_{\infty}$ control [23], adaptive control [9]. In particular, $H_{\infty}$ active suspensions have been intensively discussed in the context of robustness and disturbance attenuation [2], and they have been well recognized to be an effective way to manage the tradeoff between conflicting performance requirements. Therefore, in recent years, much attention has been devoted to the $H_{\infty}$ control of active suspensions, and a number of important results have been reported, see for example, [3], [14], [20], [22], and the references therein.

As is well known, computers are usually used as digital controllers to control continuous-time systems in modern control systems [4]. In such a system, a digital computer is used to sample and quantize a continuous-time measurement signal to produce a discrete-time signal, and then produce a discrete-time control input signal, which is further converted back into a continuous-time control input signal using a zero-order hold. Such control systems involve both continuous-time and discrete-time signals in the continuous-time framework and are referred to as sampled-data systems. Analysis and synthesis of sampled-data systems have been investigated extensively in the work of [5], [17], [19], and [21]. Among these references, there are two main approaches. One is based on the lifting technique, where the system under consideration is transformed to an equivalent finite dimensional discrete system. The other is more direct, which is based on the representation of the system in the form of hybrid discrete/continuous models and the solution is obtained in terms of differential Riccati equations with jumps. Besides these two main approaches, the continuous-time systems with digital control also can be modelled as continuous-time systems with delayed control inputs, which was introduced in [10]–[12] and [18]. In this approach, the digital control law is represented as delayed control between two sampling instants. This approach has been further developed to robust $H_{\infty}$ sampled-data control of linear systems. The most significant advantage of this input delay approach over the other two main approaches is that it does not require the sampling distances to be constant. In other words, this approach can be applied to systems with nonuniform uncertain sampling. Moreover, the input delay approach can cope with the case where system parameter uncertainties are present, which has been recognized to be a difficult problem for traditional lifting techniques.

The existing results obtained for $H_{\infty}$ control of active suspensions are mainly confined to designing continuous-time controllers for continuous-time models. When these results are applied to an actual system, we usually need to resort to digital implementation techniques. In other words, the controllers designed in the continuous-time framework need to be digitalized using some approximation methods. An interesting question would be whether we can design sampled-data controllers for active suspension systems directly. Although this seems to be a meaningful work, to the best of the authors’ knowledge, few attempts have been made towards this direction, which motivates our present study.

In this brief, the problem of $H_{\infty}$ controller design is investigated for a class of uncertain vehicle suspension systems with sampling measurements. A quarter-car model is used to study the performance of a suspension system with parameter uncer-
tainties represented by a given polytope. By using an input delay approach, the active suspension system with sampling measurements is transformed into a time-delay system, and our attention is focused on developing methods to design a state feedback control law such that the resulting closed-loop system is asymptotically stable with a prescribed level of disturbance attenuation. The desired controllers can be obtained by solving a set of linear matrix inequalities (LMIs) using standard numerical algorithms [13].

The rest of this brief is organized as follows. The problem to be solved is formulated mathematically in Section II, and controller design is presented in Section III. Section IV provides a design example and some concluding remarks are given in Section V.

Notation: The superscript “T” stands for matrix transposition; \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space and the notation \( P > 0 \ (\geq 0) \) means that \( P \) is real symmetric and positive definite (semi-definite). In symmetric block matrices or complex matrix expressions, we use an asterisk (*) to represent a term that is induced by symmetry and \( \text{diag}\{\ldots\} \) stands for a block-diagonal matrix. For a vector or matrix, \( \{\ldots\}_j \ (j = 1, 2, \ldots) \) represents the \( j \)th line of the vector or matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The space of square-integrable vector functions over \([0, \infty)\) is denoted by \( L_2[0, \infty) \), and for \( w = \{w(t)\} \in L_2[0, \infty) \), its norm is given by \( \|w\|_2 = \sqrt{\int_0^\infty |w(t)|^2 dt} \).

II. Problem Formulation

Consider the model shown in Fig. 1. In this figure, \( m_s \) is the sprung mass, which represents the car chassis; \( m_u \) is the unsprung mass, which represents mass of the wheel assembly; \( c_s \) and \( k_s \) are damping and stiffness of the suspension system, respectively; \( k_l \) and \( c_t \) stand for compressibility and damping of the pneumatic tire, respectively; \( z_s \) and \( z_u \) are the displacements of the sprung and unsprung masses, respectively; \( z_r \) is the road displacement input; \( u \) is the active input of the suspension system.

The ideal dynamic equations of the sprung and unsprung masses are given by

\[
\begin{align*}
\dot{z}_s(t) + c_s \left[ z_s(t) - z_u(t) \right] + k_s \left[ z_s(t) - z_u(t) \right] &= u(t) \\
\dot{z}_u(t) + c_u \left[ z_u(t) - z_s(t) \right] + k_u \left[ z_u(t) - z_s(t) \right] &= -u(t),
\end{align*}
\]

(1)

Define the following state variables:

\[
\begin{align*}
x_1(t) &= z_s(t) - z_u(t) \\
x_2(t) &= z_u(t) - z_r(t) \\
x_3(t) &= \dot{z}_s(t) \\
x_4(t) &= \dot{z}_u(t)
\end{align*}
\]

(2)

where \( x_1(t) \) denotes the suspension deflection, \( x_2(t) \) is the tire deflection, \( x_3(t) \) is the sprung mass speed, and \( x_4(t) \) denotes the unsprung mass speed. We define disturbance input \( w(t) = \dot{z}_r(t) \). Then, by defining \( \xi(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T \), the dynamic equations in (1) can be rewritten in the following state-space form:

\[
\dot{\xi}(t) = A(\lambda)\xi(t) + B(\lambda)w(t) + B(\lambda)u(t)
\]

(3)

where

\[
A(\lambda) = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
-k_s & 0 & c_u & 0 \\
k_s & -m_u & c_s & -c_t -c_s
\end{bmatrix},
\]

\[
B(\lambda) = \begin{bmatrix}
0 \\
0 \\
1 \\
c_t
\end{bmatrix},
\]

\[
B_1(\lambda) = \begin{bmatrix}
0 \\
-1 \\
0 \\
c_s
\end{bmatrix}
\]

where \( \lambda \) is used to characterize the parameter uncertainty, which will be described in detail subsequently.

In designing the control law for a suspension system, we need to consider ride comfort. It is widely accepted that ride comfort is closely related to the body acceleration. Therefore, when we design the controller, one of our main objectives is to reduce the body acceleration, that is, \( \ddot{z}_s(t) \).

In addition, in order to make sure the car safety, we should ensure the firm uninterrupted contact of wheels to road, and the dynamic tire load should be small, that is, \( k_t(z_u(t) - z_r(t)) < (m_s + m_u)g \).

Because of mechanical structure, the suspension stroke should not exceed the allowable maximum, that is

\[
|z_s(t) - z_u(t)| \leq z_{\text{max}}
\]

(4)

where \( z_{\text{max}} \) is the maximum suspension deflection.

Another hard constraint imposed on active suspensions is from the limited power of the hydraulic actuator

\[
|u(t)| \leq u_{\text{max}}.
\]

(5)
According to the above conditions, we choose the $H_{\infty}$ norm as performance measure and the body acceleration $z_{s}(t)$ as performance output, and choose the suspension stroke $z_{s}(t) - z_{e}(t)$ and relative dynamic tire load $k_{e}(z_{e}(t) - z_{r}(t))/(m_{a} + m_{u})g$ as constrained outputs.

Therefore, the vehicle suspension control system can be described as

\[
\begin{align*}
\dot{x}(t) &= A(\lambda)x(t) + B_{1}(\lambda)u(t) + B(\lambda)u(t) \\
\dot{z}_{1}(t) &= C_{1}(\lambda)x(t) + D_{1}(\lambda)u(t) \\
\dot{z}_{2}(t) &= C_{2}(\lambda)x(t)
\end{align*}
\]

where $A(\lambda), B_{1}(\lambda), \text{ and } B(\lambda)$ are defined in (3), and

\[
\begin{align*}
C_{1}(\lambda) &= \begin{bmatrix} -k_{s} & 0 & c_{s} & c_{a} \\ m_{a} & m_{s} \end{bmatrix} \\
D_{1}(\lambda) &= \frac{1}{m_{a}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
C_{2}(\lambda) &= \begin{bmatrix} 1 & k_{s} \\ 0 & (m_{a} + m_{u})g \end{bmatrix}
\end{align*}
\]

By considering the modelling uncertainty, in this brief we assume the matrices $A(\lambda), B_{1}(\lambda), B_{2}(\lambda), C_{1}(\lambda), D_{1}(\lambda),$ and $C_{2}(\lambda)$ in (6) contain uncertain parameters, represented by $\lambda$. It is assumed that $\lambda$ varies in a polytope of vertices $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{r}$, i.e., $\lambda \in \Theta \coloneqq Co[\lambda_{1}, \lambda_{2}, \cdots, \lambda_{r}]$, where the symbol $Co$ denotes the convex hull, and thus we have

\[
\Theta \triangleq \{ \lambda \mid \lambda = \sum_{i=1}^{r} \lambda_{i} \Omega_{i}; \sum_{i=1}^{r} \lambda_{i} = 1, \lambda_{i} \geq 0 \}
\]

where $\Theta$ is a given convex bounded polyhedral domain described by $r$ vertices

\[
\Omega_{i} \triangleq \{ \Omega \mid \Omega = \sum_{i=1}^{r} \lambda_{i} \Omega_{i}; \sum_{i=1}^{r} \lambda_{i} = 1 \}
\]

with $\Omega_{i} \triangleq (A_{i}, B_{1i}, B_{2i}, C_{1i}, C_{2i}, D_{1i})$ denoting the vertices of the polytope.

It is assumed that the state variables of the active suspension system are measured at time instants $t_{k}, \ldots, t_{k+1}, \ldots$ that is, only $x(t_{k})$ are available for interval $t_{k} \leq t < t_{k+1}$. We are interested in designing a state feedback controller of the form

\[
u(t) = u(t_{k}) = Kx(t_{k}), \quad t_{k} \leq t < t_{k+1}
\]

where $K$ is the state feedback gain matrix to be designed.

Therefore, the closed-loop system is given by

\[
\begin{align*}
\dot{x}(t) &= A(\lambda)x(t) + B_{1}(\lambda)u(t) + B(\lambda)Kx(t_{k}) \\
\dot{z}_{1}(t) &= C_{1}(\lambda)x(t) + D_{1}(\lambda)Kx(t_{k}) \\
\dot{z}_{2}(t) &= C_{2}(\lambda)x(t), \quad t_{k} \leq t < t_{k+1}.
\end{align*}
\]

(10)

It is assumed that $w \in L_{2}[0, \infty)$, and without loss of generality, we have $\|w\|_{2} \leq \bar{u}_{\text{max}} < \infty$. Then, the objective of this brief is to determine a controller gain $K$ such that:

1) the closed-loop system is asymptotically stable;
2) under zero initial condition, the closed-loop system guarantees that $\|z_{1}\|_{2} < \gamma\|w\|_{2}$ for all nonzero $w \in L_{2}[0, \infty)$, where $\gamma > 0$ is a prescribed scalar;
3) the following control output and input constraints are guaranteed:

\[
\begin{align*}
\|z_{2}(t)\|_{2} &\leq \|z_{2,\text{max}}\|_{2}, \quad j = 1, 2, \|u(t)\| \leq \bar{u}_{\text{max}}, t > 0 \quad (11)
\end{align*}
\]

where $z_{2,\text{max}}$ is a prescribed scalar.

Before proceeding further, we first introduce the following general assumption.

**Assumption 1**: It is assumed that the interval between any two sampling instants is bounded by $h$ ($h > 0$). That is, $t_{k+1} - t_{k} \leq h, \forall k \geq 0$. The closed-loop system in (10) involves both continuous and discrete signals. Due to parameter uncertainties, it is difficult to use the traditional lifting techniques to solve this sampled-data control problem.

**Remark 2**: It is known that ride comfort is frequency sensitive. Although our brief considers the ride comfort in full frequency, it is worth mentioning that our approach can be further extended to finite frequency case, by incorporating some frequency weighting transfer function.

### III. CONSTRAINED SAMPLED-DATA CONTROLLER DESIGN

In this section, the problem formulated above will be solved by an input delay approach. The key idea behind this approach is that we represent the sampling instant $t_{k}$ as

\[
t_{k} = t - (t - t_{k}) = t - \bar{d}(t)
\]

where $\bar{d}(t) = t - t_{k}$. Then, we obtain

\[
u(t) = u(t_{k}) = u(t - \bar{d}(t)), \quad t_{k} \leq t < t_{k+1}
\]

where $u(t_{k})$ is a discrete-time control signal and the time-varying delay $\bar{d}(t)$ is piece-wise linear with derivative $\dot{\bar{d}}(t) = 1$ for $t \neq t_{k}$. Recently, the $H_{\infty}$ control problem for active vehicle suspension systems with input delay has been addressed in [8], where the time-delay is fixed and constant. It is worth pointing out that the transformed system in our problem contains non-differentiable-time-varying delay in the states, which hinders the results in [8] to be directly applied to the problem considered here.

By making use of (12), the sampled-data formulation in (10) can be transformed into the following system:

\[
\begin{align*}
\dot{x}(t) &= A(\lambda)x(t) + B(\lambda)Kx(t - \bar{d}(t)) + B_{1}(\lambda)u(t) \\
\dot{z}_{1}(t) &= C_{1}(\lambda)x(t) + D_{1}(\lambda)Kx(t - \bar{d}(t)) \\
\dot{z}_{2}(t) &= C_{2}(\lambda)x(t),
\end{align*}
\]

(13)

Now, a continuous-time system with a time-varying delay $\bar{d}(t)$ in the state, as shown in (13), has been obtained by transforming the sampled-data closed-loop system in (10) as above. In the following, we will investigate how to design a desired sampled-data controller based on the transformed closed-loop system in (13).

**Theorem 1**: Consider the active suspension system in (10) under Assumption 1. Given scalars $\gamma > 0, h > 0,$ and $\rho > 0$, if there exist matrices $P = P^{T} > 0, Q = Q^{T} > 0,$ and $S_{i}$
satisfying
\[
\begin{bmatrix}
\Psi_{1i} + \Psi_{2i} + \Psi_{5i} & \sqrt{h} \Phi^{T}_{1i} & \sqrt{h} S_{i} & \Phi^{T}_{2i} \\
* & * & -Q & 0 \\
* & * & -Q & 0 \\
* & * & -Q & 0 \\
\end{bmatrix}
< 0, \quad i = 1, \cdots, r
\]
(14)
\[
\begin{bmatrix}
-I & \sqrt{p[C_{2k}]_{ij}} & 0 & 0 \\
* & \{2 \pi_{2k} \}_{ij} & 0 & 0 \\
-I & \sqrt{pK} & 0 & 0 \\
* & -\frac{\lambda_{2}}{\max} P & 0 & 0 \\
\end{bmatrix}
< 0, \quad i = 1, \cdots, r, \quad j = 1, 2
\]
(15)
\[
\begin{bmatrix}
-I & \sqrt{pK} \\
* & -\frac{\lambda_{2}}{\max} P \\
\end{bmatrix}
< 0
\]
(16)
where
\[
\Psi_{1i} = \begin{bmatrix}
PA_{i} + A^{T}_{i} P & PB_{1i} & PB_{1i}^{T} & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\Psi_{2i} = \begin{bmatrix}
S_{i} & -S_{i} & 0 \\
0 & 0 & -\gamma \; I \\
\end{bmatrix}
\]
\[
\Psi_{5i} = \text{diag} \begin{bmatrix}
0 & 0 & -\gamma^{2} I \\
\end{bmatrix}
\]
\[
\Phi_{1i} = [A_{i}, \; B_{i} K \; B_{1i}^{T}] \\
\Phi_{2i} = [C_{2i}, \; D_{2i} K \; 0]
\]
(17)
then a stabilizing controller in the form of (9) exists, such that:
1) the closed-loop system is asymptotically stable;
2) under zero initial condition, the closed-loop system guarantees that \( \|z(t)\|_{2} < \gamma \|w\|_{2} \) for all nonzero \( w \in L_{2}[0, \infty) \);
3) the control output and input constraints (11) are guaranteed with the disturbance energy under the bound \( u_{\max} = (\rho - V(0))/\gamma^{2} \).

**Proof:** Under the condition of theorem, first, we show the asymptotic stability of (13) with \( u(t) = 0 \), that is
\[
\dot{x}(t) = A(\lambda)x(t) + B(\lambda)Kx(t - d(t)).
\]
(18)
Now, choose a Lyapunov functional candidate for system (18) as
\[
V(t) = x^{T}(t)Px(t) + \int_{-h}^{t} \int_{-h}^{\tau} \dot{x}(\alpha)Q\dot{x}(\alpha)d\alpha d\beta
\]
(19)
where \( P > 0 \) and \( Q > 0 \) are matrices to be determined.
The derivative of \( V(t) \) satisfies
\[
\dot{V}(t) = 2x^{T}(t)P\dot{x}(t) + h\dot{x}^{T}(t)Q\dot{x}(t) - \int_{t-h}^{t} \dot{x}(\alpha)Q\dot{x}(\alpha)d\alpha,
\]
In addition, by the Newton–Leibniz formula, for any appropriately dimensioned matrices \( \dot{S}(\lambda) = \sum_{i=1}^{r} \lambda_{i} \dot{S}_{i} \), we have
\[
\dot{S}^{T}(t) \dot{S}(\lambda) \left[ x(t) - x(t - d(t)) \right] - \int_{t-h}^{t} \dot{x}(\alpha)d\alpha = 0
\]
where \( \zeta(t) = [x^{T}(t) x^{T}(t - d(t))]^{T} \). So we can get
\[
\dot{V}(t) \leq 2x^{T}(t)P[A(\lambda)x(t) + B(\lambda)Kx(t - d(t))]
\]
\[
- \int_{t-h}^{t} \dot{x}(\alpha)Q\dot{x}(\alpha)d\alpha,
\]
(20)
where
\[
\dot{S}(\lambda) + \dot{S}^{T}(\lambda)\zeta(t) + \dot{\zeta}(\alpha)Q^{-1}\dot{\zeta}(\alpha) = 0
\]
(21)
Then, the time derivative of \( V(t) \) along the solution of system (18) holds
\[
\dot{V}(t) \leq \zeta^{T}(t) \left[ E_{1}(\lambda) + E_{2}(\lambda) + E_{3}(\lambda) + E_{4}(\lambda) \right] \zeta(t)
\]
\[
+ h\dot{S}(\lambda)Q^{-1}\dot{S}^{T}(\lambda)\zeta(t)
\]
\[
- \int_{t-h}^{t} \left[ \dot{\zeta}(t)\dot{S}(\lambda) + \dot{\zeta}(\alpha)Q^{-1}\dot{\zeta}(\alpha) \right] d\alpha
\]
(22)
Note that \( Q > 0 \), thus \( \int_{t-h}^{t} \dot{\zeta}^{T}(t)\dot{S}(\lambda) + \dot{\zeta}^{T}(\alpha)Q^{-1}\dot{\zeta}(\alpha) d\alpha < 0 \) is positive. By Schur complement, inequality (14) guarantees
\[
E_{1}(\lambda) + E_{2}(\lambda) + E_{3}(\lambda) + h\dot{S}(\lambda)Q^{-1}\dot{S}^{T}(\lambda) < 0
\]
(23)
According to the inner property of polytopic uncertain systems, and considering the form \( A(\lambda) = \sum_{i=1}^{r} \lambda_{i}A_{i} \), \( B(\lambda) = \sum_{i=1}^{r} \lambda_{i}B_{i} \), \( \dot{S}(\lambda) = \sum_{i=1}^{r} \lambda_{i}\dot{S}_{i} \), from (22) we obtain
\[
E_{1}(\lambda) + E_{2}(\lambda) + E_{3}(\lambda) + h\dot{S}(\lambda)Q^{-1}\dot{S}^{T}(\lambda) < 0,
\]
(23)
Therefore, we have \( \dot{V}(t) < 0 \), and the asymptotic stability is established.
Next, we shall establish the \( H_{\infty} \) performance of the system in (13) under zero initial conditions. Firstly, define the Lyapunov
It is also true that: $x^T(t-d(t))P_x(t-d(t)) < \rho$, with $t > d(t)$. Consider
\[
\max_{t \geq 0} \left\{ z_{2i}(t) \right\}_i^2 = \max_{t \geq 0} \left\| x^T(t)\{C_{2i}\}^T \{C_{2i}\}^j x(t) \right\|_2
\leq \max_{t > d(t)} \left\| x^T(t)P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j \right\|_2 \times P^{-\frac{1}{2}} \frac{1}{2} x(t) \right\|_2 < \rho \cdot \theta_{\text{max}} \left( P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j P^{-\frac{1}{2}} \right), \quad i = 1, \ldots, r, \quad j = 1, 2,
\]
where $\theta_{\text{max}}(\cdot)$ represents maximal eigenvalue. From the above inequalities, we know that the constraints (11) are guaranteed, if
\[
\rho \cdot P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j P^{-\frac{1}{2}} < \{z_{2i,\text{max}}\}_i^2 I,
\]
where $\theta_{\text{max}}(\cdot)$ represents maximal eigenvalue. From the above inequalities, we know that the constraints (11) are guaranteed, if
\[
\rho \cdot P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j P^{-\frac{1}{2}} < \{z_{2i,\text{max}}\}_i^2 I,
\]
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\]
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\]
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\]
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\rho \cdot P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j P^{-\frac{1}{2}} < \{z_{2i,\text{max}}\}_i^2 I,
\]
where $\theta_{\text{max}}(\cdot)$ represents maximal eigenvalue. From the above inequalities, we know that the constraints (11) are guaranteed, if
\[
\rho \cdot P^{-\frac{1}{2}} \{C_{2i}\}^T \{C_{2i}\}^j P^{-\frac{1}{2}} < \{z_{2i,\text{max}}\}_i^2 I,
then a stabilizing controller in the form of (9) exists, such that
1) the closed-loop system is asymptotically stable;
2) under zero initial condition, the closed-loop system guarantees that \(|z_1| < \gamma |w|\) for all nonzero \(w \in L_2[0, \infty)\);
3) the control output and input constraints (11) are guaranteed with the disturbance energy under the bound \(w_{\text{max}} = (\rho - V(0))/\gamma^2\).

Moreover, if inequalities (31)–(33) have a feasible solution, then the control gain \(K\) in (9) is given by

\[K = KP^{-1}.\] (35)

Remark 3: The conditions in Theorem 2 are LMIs not only over the matrix variables, but also over the scalar \(\gamma\). This implies that the scalar \(\gamma\) can be included as an optimization variable to obtain a reduction of the guaranteed \(H_{\infty}\) performance bound. Then the minimal \(\gamma\) can be found by solving the following convex optimization problem: minimize \(\gamma\) subject to (31)–(33) over \(P > 0, \bar{Q} > 0, \mathcal{S}_i > 0\), and \(\bar{K}\).

IV. DESIGN EXAMPLE

In this section, we provide an example to illustrate the effectiveness of the proposed sampled-data \(H_{\infty}\) controller design method. The quarter-car model parameters are borrowed from [8] and listed in Table I.

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>QUARTER-CAR MODEL PARAMETERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_s)</td>
<td>(m_u)</td>
</tr>
<tr>
<td>973 kg</td>
<td>114 kg</td>
</tr>
</tbody>
</table>

First, consider the nominal system, whose parameter matrices have no uncertainties. Assume the maximum allowable suspension stroke \(z_{\text{max}} = 0.08\) m, the maximum force output \(u_{\text{max}} = 1500\) N, the sampling interval \(h = 10\) ms. Here, we choose \(\rho = 1\) (the detailed discussion of its selection is given in [2]). By solving the convex optimization problem formulated in the above section, the minimum guaranteed closed-loop \(H_{\infty}\) performance obtained is \(\gamma_{\text{min}} = 8.6758\). Then, an admissible control gain matrix is \(K = 10^3 \times [0.7646 3.6362 - 5.3292 - 0.0046]\).

In the following, we will illustrate the performance of the closed-loop sampled-data suspension system. Evaluation of the vehicle suspension performance is based on the examination of three response quantities, that is, the sprung mass acceleration \(z_1(t)\), the suspension deflection and the tire deflection, which can be shown from \(z_2(t)\). A controller is to be designed such that: 1) the sprung mass acceleration \(z_1(t)\) is as small as possible; 2) the suspension deflection is below the maximum allowable suspension stroke \(z_{\text{max}} = 0.08\) m; 3) the controlled output defined in (6) satisfy \(\{z_2(t)\} < 1\); and 4) the active force \(|u(t)| \leq u_{\text{max}}\) In order to evaluate the suspension characteristics with respect to ride comfort, vehicle handling, and working space of the suspension, the variability of the road profiles is...
taken into account. In the context of vehicle suspension performance, road disturbances can be generally assumed as shock. Shocks are discrete events of relatively short duration and high intensity, caused by, for example, a pronounced bump or pot-hole on an otherwise smooth road. In the following, a kind of road profile is used to validate the performance of the presented control approach.

Now consider the case of an isolated bump in an otherwise smooth road surface [15]. The corresponding ground displacement is given by

$$z_e(t) = \begin{cases} \frac{A}{2} \left( 1 - \cos \left( \frac{2\pi V}{L} t \right) \right), & \text{if } 0 \leq t \leq \frac{L}{V} \\ 0, & \text{if } t > \frac{L}{V} \end{cases}$$

(36)

where $A$ and $L$ are the height and the length of the bump. Assume $A = 60$ mm, $L = 5$ m and the vehicle forward velocity as $V = 45$ (km/h). In order to compare, we give the controller of continue-time system which is obtained from [8]

$$K_c = 10^4 \times \begin{bmatrix} -8.9220 & -0.1447 & -3.6650 & 0.1491 \end{bmatrix}.$$  

The responses of the open-loop system ($u(t_k) = 0$, passive mode), the continue-time closed-loop system ($u(t) = K_c x(t)$) and the sampled-data closed-loop system (active mode) which is composed by the controller we designed above are depicted in Fig. 2, which shows the bump response of the body acceleration, the suspension deflection, the tire deflection, and the active force, respectively, with the sampling interval $h = 10$ ms, where the passive suspension, the continue-time active suspension and sampled-data active suspension are depicted in point lines, point-solid lines and solid lines, respectively. From these figures, we can see that the sampled-data controller yields the least value of the maximum body acceleration, compared with the open-loop system and the continuous-time controller. In addition, we can see that the active control force constraint is respected by the sampled-date control, while not respected by the continuous-time controller due to its ignorance of the hard constraints in the controller design process.

It is interesting to note that the sampling period has much to do with the guaranteed performance $\gamma_{\min}$ we obtain by the convex optimization problem formulated in the above section. Table II lists the guaranteed performance $\gamma_{\min}$ we obtain for different sampling periods, from which we can see that the guaranteed performance is larger when the sampling period increases.

Now, we consider the uncertain case, that is, we design robust sampled-data controllers for uncertain suspension systems. Assume that the sprung mass $m_s$ and the unsprung $m_u$ contain uncertainties, which are expressed as $m_s = (973 + \lambda_1)$ kg, $m_u = (114 + \lambda_2)$ kg, where $\lambda_1$ and $\lambda_2$ satisfy $|\lambda_1| \leq \dot{\lambda}_1$ and $|\lambda_2| \leq \dot{\lambda}_2$. It is assumed that $\dot{\lambda}_1 = 100$ and $\dot{\lambda}_2 = 10$. In this case, the suspension system can be represented by a four-vertex polytopic system. By solving the corresponding convex optimization problem, when the sampling period $h = 10$ ms, the minimum guaranteed closed-loop $H_{\infty}$ performance obtained is $\gamma_{\min} = 10.4220$. An admissible robust control gain matrix is given by $K = 10^3 \times \begin{bmatrix} 1.3308 & 3.9564 & -5.5819 & 0.5438 \end{bmatrix}$.

In the following, we will illustrate the performance of the closed-loop sampled-data suspension system with parameter uncertainties. Fig. 3 depicts the bump response of the body
acceleration, and the constrained conditions (suspension deflection, tire deflection and active force) with the sampling interval $h = 10$ ms. In each figure, the four vertex systems are depicted with the sampling interval $h = 10$ ms. The effectiveness of the control design is apparent from these figures.

V. CONCLUDING REMARKS

In this brief, the problem of robust sampled-data $H_{\infty}$ control for uncertain active vehicle suspension systems has been investigated. By using an input delay approach, the active vehicle suspension system with sampling measurements has been transformed into a continuous-time system with a delay in the state, and polytopic parameter uncertainty has been utilized to characterize the real uncertain situation. A quarter-car model has been considered and the effectiveness of the proposed approach has been illustrated by a practical design example.

REFERENCES
