Static Output Feedback Stabilization: An ILMI Approach*

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Key Words—Static output feedback; linear matrix inequality (LMI); stabilization; linear time-invariant systems.

Abstract—In this note, the static output feedback stabilization problem is addressed using the linear matrix inequality technique. A necessary and sufficient condition for static output feedback stabilizability for linear time-invariant systems is derived in the form of a matrix inequality. The extension of the result to $H_\infty$ control is studied. An iterative LMI (ILMI) algorithm is proposed to compute the feedback gain. Numerical examples are employed to demonstrate the effectiveness and the convergence of the algorithm. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

The static output feedback problem is one of the most researched problems in control theory and applications (see recent surveys by Bernstein, 1992; Blondel et al., 1995; Syrmos et al., 1997). Despite the efforts over the last decade, there still remain many important problems to be solved analytically or numerically. In its simplest form associated with a finite-dimensional linear time-invariant system to be controlled, the static output feedback problem concerns finding a static or constant feedback gain to achieve certain desirable characteristics. One reason why static output feedback has received so much attention is that it is the simplest closed-loop control that can be realized in practice. Another important reason is that many problems involving synthesizing dynamic controllers can be formulated as static output feedback control problems involving augmented plants. A variety of static output feedback problems were studied by many researchers with many analytical and numerical methods proposed (see Barmish, 1985; Bernstein, 1992; Blondel et al., 1995; Gahinet and Apkarian, 1994; Gromov, 1995; Gu, 1990; Iwasaki and Skelton, 1995; Iwasaki et al., 1994; Kimura, 1975; Kucera and de Souza, 1995; Oliveira and Gerome, 1997; Perez et al., 1993; Schumacher, 1980; Syrmos et al., 1997 and references therein).

Stability is the most important characteristics to be maintained in almost all practical control systems. Stabilization using static output feedback has been recently studied by, amongst others, Bernstein (1992), Iwasaki and Skelton (1995), Toker and Ozbay (1995), and Syrmos et al. (1997), etc. Very often, the algebraic necessary and sufficient conditions for output feedback stabilization derived are not readily implementable as numerical algorithms. Another major difficulty is due to the non-convexity of the static output feedback solution set, which renders it a non-trivial computational task, analytical and numerical alike. These motivate the present work to present a necessary and sufficient condition for static output feedback stabilization using the matrix inequality approach.

2. Static output feedback stabilization

In this note, $X > 0$ (resp. $X \geq 0$) means that $X$ is a real symmetric and positive-definite (resp. semidefinite) matrix. All matrices are with compatible dimensions if they are not explicitly stated.

Consider a finite-dimensional linear time-invariant system $\Sigma$ described by

$$\Sigma : \dot{x} = Ax + Bu, y = Cx,$$

(1)

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is the control vector, and $y \in \mathbb{R}^m$ is the output vector, $A$, $B$, and $C$ are constant matrices. $\Sigma$ may also be identified by its realization $(A, B, C)$. All systems considered in this note are assumed to be stabilizable via state feedback. The static output feedback stabilization problem is to find a static output feedback $u = Fy$, where $F \in \mathbb{R}^{r \times m}$, such that the closed-loop system $\Sigma_F$ given by

$$\Sigma_F : \dot{x} = (A + BF)x,$$

is stable, that is, with poles in the open left-half-plane.

Definition 1. $\Sigma$ is said to be stabilizable via static output feedback if there exists $F$ such that $\Sigma_F$ is stable, $\Sigma$ is said to be $\Sigma$-stabilizable via static output feedback if $F$ places the closed-loop poles to the left of a vertical line $\text{Re}(s) = -\alpha$ (for some real $\alpha$) in the complex plane.

Theorem 1. $\Sigma$ is stabilizable via static output feedback if and only if $P > 0$ and $F$ satisfying the following matrix inequality:

$$A^T P + PA - PBB^T P + (B^T P + FC)(B^T P + FC)^T < 0.$$ (2)

Proof. Sufficiency. Note that

$$(A + BF)^T P + P(A + BF)$$

$$\leq (A + BF)^T P + P(A + BF) + C^TF FC$$

$$= A^T P + PA - PBB^T P + (B^T P + FC)(B^T P + FC)$$

$$< 0.$$

From Lyapunov's stability theorem, we know that $\Sigma_F$ is stable.

To prove necessity, suppose $\Sigma_F$ is stable for some $F$. Then there exists $P > 0$ such that

$$(A + BF)^T P + (A + BF) < 0.$$
It is easy to find there exists a scalar \( \rho > 0 \) such that
\[
(A + BFC)^TP + P(A + BFC) + \frac{1}{\rho^2}C^TF^FC < 0,
\]
i.e.
\[
\rho^2 A^TP + PA - \rho^3PBB^TP + (\rho^2B^TP + FC)^T
\times (\rho^2B^TP + FC) < 0.
\]
(3)

Obviously, equation (2) is equivalent to
\[
\rho^2 A^TP + PA - \rho^3PBB^TP + (\rho^2B^TP + FC)^T
\times (\rho^2B^TP + FC) < 0.
\]
By substituting \( \rho^2P \) with \( P \), we obtain inequality (2). \( \square \)

This theorem is similar to the result of Kucera and de Souza (1995), in which the main result is stated in the form of algebraic Riccati equation. Since concepts such as stabilizable and detectable are not explicitly employed in Theorem 1, the proof is much more transparent. Moreover, the present result is given as a QMI and is more suitable for the numerical procedure to be developed in this work. This theorem can also be proved using the approach of Iwasaki and Skelton (1995).

**Corollary 1.** \( \Sigma \) is \( \sigma \)-stabilizable via static output feedback if and only if there exist two matrices \( P > 0 \) and \( F \) satisfying the following matrix inequality:
\[
A^TP + PA - PBB^TP + (B^TP + FC)^T(B^TP + FC) - zP < 0.
\]
(4)

By duality, we obtain another result.

**Corollary 2.** \( \Sigma \) is stabilizable via static output feedback if and only if there exist two matrices \( Q > 0 \) and \( F \) satisfying the following matrix inequality:
\[
QA^T + AQ - QC^TCQ + (BF + CQ)(BF + CQ)^T < 0.
\]

3. Iterative LMI algorithm

Note that matrix inequality (2) is a QMI. If it is possible to find \( P > 0 \) and \( F \) satisfying the QMI in equation (2), then a stabilizing static output feedback gain exists. An advantage of this approach to obtain a stabilizing feedback gain \( F \) is no longer assumed to be a function of the solution \( P \) of a special equation or inequality. For this reason, the approach can be applied to other problems such as simultaneous stabilization, and decentralized stabilization.

Due to the negative sign in the \(-PBB^TP\) term, equation (2) cannot be simplified to an LMI. To accommodate the \(-PBB^TP\) term, we introduce an additional design variable \( X \). Because \( (X - P)^TB^TX - P > 0 \) for any \( X \) and \( P \) of the same dimension, we obtain
\[
X^TB^TP + X^TBB^TP - X^TBB^TX \leq X^TB^TP
\]
with equality holds when \( X = P = 0 \). By combining equations (5) and (2), we obtain a sufficient condition for the existence of static output feedback gain matrix \( F \) given by
\[
A^TP + PA - XBB^TP - PBB^TX + XBB^TX + (B^TP + FC)^T(B^TP + FC) < 0.
\]
(6)

**Theorem 2.** \( \Sigma \) is stabilizable via static output feedback if and only if there exist \( P > 0 \) and \( F \) satisfying the matrix inequality in equation (6).

**Proof.** The sufficiency is obvious, only the necessity needs to be proven. Suppose \( \Sigma \) is stabilizable via static output feedback, then there exist \( P > 0 \) and \( F \) such that
\[
A^TP + PA - PBB^TP + (B^TP + FC)^T(B^TP + FC) < 0,
\]
then there exists a number \( \epsilon > 0 \) such that
\[
A^TP + PA - PBB^TP + (B^TP + FC)^T(B^TP + FC) + \epsilon \leq 0.
\]
Select a \( \bar{X} \geq BB^T \) and set \( X = P - \Delta X \) where \( \Delta X = \epsilon^{1/2}X^{-1/2} \), then
\[
(P - X)BB^T(X - P) \leq \epsilon.
\]
Hence, equation (6) holds.

Using the Schur complement, inequality (6) is equivalent to the following quadratic matrix inequality:
\[
\begin{bmatrix}
A^TP + PA - XBB^TP - PBB^TX + XBB^TX & (B^TP + FC)^T(B^TP + FC)

(B^TP + FC)

- I
\end{bmatrix} < 0.
\]
(7)

This QMI points to an iterative approach to solve \( F \) and \( P > 0 \), namely, if \( X \) is fixed in equation (7), then it reduces to an LMI problem in the unknowns \( F \) and \( P \). The LMI problem is convex and can be solved efficiently if a feasible solution exists.

Even \( X \) is fixed, however, LMI (7) is only a sufficient condition for stabilizability via static output feedback. In fact, if we find a solution of LMI (7), then we find a solution of equation (2).

But, in general, it has no solution for a fixed \( X \). On the other hand, if we simply perturb \( A \) to \( A - (\alpha/2)I \) for some \( \alpha \geq 0 \), we obtain a necessary condition for the feasibility of equation (2). That is, if the matrix inequality (7) has a solution \( (P > 0, F) \), then there exist a real number \( \alpha \geq 0 \) and a matrix \( X > 0 \) such that
\[
A^TP + PA - \alpha P - XBB^TP - PBB^TX + XBB^TX + (B^TP + FC)^T(B^TP + FC) < 0.
\]
Consequently, the closed-loop system matrices \( A - BFC \) have eigenvalues on the left-hand side of the line \( \Re(\lambda) = \alpha \) in the complex \( \lambda \)-plane. Based on the idea that all eigenvalues of \( A - BFC \) are shifted progressively towards the left-half-plane through the reduction of \( \alpha \), we may close in on the feasibility of equation (2).

**Iterative linear matrix inequality (ILMI) algorithm**

Given \( \Sigma \) with realization \((A, B, C)\) stabilizable via state feedback.

**Step 1.** Select \( Q > 0 \), and solve \( P \) from the following algebraic Riccati equation
\[
A^TP + PA - PBB^TP + Q = 0.
\]
Set \( i = 1 \) and \( X_1 = P \).

**Step 2.** Solve the following optimization problem for \( P_i, F \) and \( z_i \).

**OP1:** Minimize \( z_i \) subject to the following LMI constraints
\[
\begin{bmatrix}
A^TP_i + P_iA - X_iB^TP_i - P_iBB^TX_i + X_iBB^TX_i - z_iP_i

(B^TP_i + FC)^T

(B^TP_i + FC)

- I
\end{bmatrix} < 0,
\]
(8)

\[ P_i = P_i^T > 0. \]
(9)

Denote \( z_\ast \) as the minimized value of \( z_i \).

**Step 3.** If \( z_\ast \leq 0 \), \( P \) is a stabilizing static output feedback gain. Stop.

**Step 4.** Solve the following optimization problem for \( P_i \) and \( F \).

**OP2:** Minimize \( \text{trace}(P_i) \) subject to the above LMI constraints (8) and (9) with \( z_i = z_\ast \). Denote \( P_\ast \) as the \( P_i \) that minimized trace \( \text{trace}(P_i) \).

**Step 5.** If \( |X_i - P_\ast| < \delta \), a prescribed tolerance, go to Step 6.

**Step 6.** The system may not be stabilizable via static output feedback. Stop.
The optimization problem OP1 in Step 2 is a generalized eigenvalue minimization problem. This step guarantees the progressive reduction of $\lambda_i$.

Inequality (7) plays a crucial role in the ILMI algorithm. Obviously, there must exist a solution for the optimization problem OP1 when $i = 1$. For $i > 1$, the existence of the solution is guaranteed by inequality (7). For a given system stabilizable via static output feedback, that is, inequality (7) has a solution, the solution sequence $\lambda_i^*$ is a decreasing sequence. This is because if the inequality

$$A^*P_i + PA_i - X_iBB_i^TP_i - P_iBB_iX_i + X_iBB_i^TP_i + (B_iP_i + FC_i)(B_iP_i + FC_i) < 0$$

holds for $P_i > 0$, then

$$A^*P_i + PA_i - X_iBB_i^TP_i - P_iBB_iX_i + X_iBB_i^TP_i + (B_iP_i + FC_i)(B_iP_i + FC_i) < 0.$$ 

In other words, a solution $\lambda_i^* \leq \lambda_i^*$ can be found in Step 2 since equation (8) is feasible with $P_{i-1} = P_i^*$, $x_{i-1} = x_i$. However, sometimes a smaller $x_{i-1}$ is not obtained in this step, that is $\lambda_i^* > \lambda_i^*$, which is due to the implementation problem of the LMI program. In this situation, we set $x_i = x_{i-1} + \Delta x_i$ for some $\Delta x_i > 0$, and solve OP2 again.

Step 3 is to guarantee the convergence of the algorithm. The existence of the solution to the optimization problem OP2 is guaranteed by inequality (8). The solution $P_i^* > 0$ implies that the sequence $\{P_i^*\}$ is bounded below. It is not difficult to observe that the solution sequence $\{P_i^*\}$ is a monotonic decreasing sequence for a fixed $x_i$ for $i > k$, where $k$ is a positive constant. This fact is discussed in many literature, see, e.g. Boyd et al. (1994) and Gahinet et al. (1995). As a result, one concludes that this ILMI algorithm is convergent although we may not achieve the solution because $x_i$ may not always converge to its minimum.

Note that the initial choice of $Q$ will affect the convergence of the algorithm because of the following reason. Assume $\tilde{F}$ is a stabilizing feedback gain, then there exists a matrix $P > 0$ such that

$$(A + BFC)P + P(A + BFC) + Q_0 = 0,$$

where $Q_0 > 0$. If an initial $Q = (BFC)P + PBC + Q_0$ is selected, then the algorithm will converge immediately. Therefore, if the algorithm fails to arrive at a stabilizing solution, we may select another $Q$ and run the ILMI algorithm again. Our numerical experience indicates that the initial choice with $Q = I$ always leads to a convergent result.

4. Extension to static output feedback $H_\infty$ control

Consider the linear time-invariant continuous-time dynamical systems with the following state-space representation:

$$\dot{x} = Ax + B_1w + B_2x,$$
$$z = C_1x + D_{11}w + D_{12}w,$$
$$y = C_2x + D_{21}w.$$ 

The static output feedback $H_\infty$ control problem is to find a static output feedback $u = Fz$ such that the closed-loop transfer function from $w$ to $z$ is stable and

$$\|T_{zw}\|_\infty < \gamma.$$ 

Under certain assumptions on the matrices in equation (10) and from Bounded Real lemma, we have $F$ is an $H_\infty$ controller if and only if there exists $P > 0$ such that

$$A_{CL}^*F + FA_{CL} + PBB_{CL}C_{CL}^* + B_{CL}^*F - \gamma I < 0,$$
$$C_{CL}^*D_{CL} - \gamma I < 0,$$
$$PBB_{CL}C_{CL}^* + \gamma I < 0.$$

where

$$A_{CL} = A + B_2FC_2,$$
$$B_{CL} = B_1 + B_2FD_{21},$$
$$C_{CL} = C_1 + D_{11}FC_2,$$
$$D_{CL} = D_{11} + D_{12}FD_{21},$$

i.e.

$$PBB_{CL}C_{CL}^* + \gamma I < 0,$$ 

$$P < 0,$$

$$A_{CL} = \begin{pmatrix} A & B_1 & 0 \\ 0 & C_1 & D_{11} \\ 0 & 0 & \gamma I \end{pmatrix},$$

$$B = \begin{pmatrix} B_2 \\ 0 \\ D_{12} \end{pmatrix},$$

$$C = [C_2 \ D_{21} \ 0].$$

Hence, the static output feedback $H_\infty$ control problem is reduced to finding $P > 0$ and $F$ such that matrix inequality (12) holds, which corresponds a generalized static output feedback stabilization problem of the system $(A, B, C)$. This problem can thus be solved using the iterative LMI approach introduced in Section 3.

5. Numerical examples

Example 1. Consider $\Sigma$ with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$C = [1 \ \beta]$$

which is unstable since its eigenvalues are 1 and $-1$. It is not difficult to show that $\Sigma$ is stabilizable via static output feedback even if $\beta > 0$. At $\beta = 15$, using the ILMI Algorithm, we obtained $x = -0.0377$ after 15 iterations and

$$\gamma = 0.0377, F = -0.7369.$$ 

The closed-loop system $\Sigma$ has eigenvalues $-0.3684 \pm 3.1492i$ and hence is stable. The algorithm due to Kucera and de Souza (1995), however, has led to a divergent result. For $\beta = 0$, we obtain $x = 5.38367 - 009 > 0$ after 11 iterations, indicating that $\Sigma$ may not be stabilizable via static output feedback. At the last iteration, the eigenvalues of $\Sigma$ are $-6.85316 + 008$ and $0$. For $\beta = -15$, we obtained $x = 1.0504 > 0$ after 77 iterations.

Example 2. To demonstrate the effectiveness and the convergence of the algorithm, we consider the numerical experiment of Oliveira and Geromel (1997). In their paper, several well-known numerical methods to find static output feedback gain were compared. It was found that the product reduction algorithm (PRA) of El Ghaoui et al. (1999) and the min-max algorithm (MMA) of Geromel et al. (1996) are the best two algorithms. As in Oliveira and Geromel (1997), we have randomly generated systems $(A, B, C)$ with two characteristics:

(a) stabilizable via static output feedback,
(b) open-loop unstable.

To generate a random static output feedback stabilizable system of given input, output and state dimensions, we first randomly generate matrices $A$, $B$, and $C$ of compatible dimensions to form a realization $(A, B, C)$. The system matrix $A$ is then made stable by reflecting the eigenvalues in the right-half-plane about the imaginary axis to the left-half-plane. In order to attain a common degree of stability equal to one, $A$ is subtracted by an appropriate scalar matrix. Then the matrix product $BFC$ is computed with a random $F$ of compatible dimensions is added such that $A = A + BFC$ to guarantee that the resulting $(A, B, C)$ is stabilizable via static output feedback. Six cases with different input, output and state dimensions are considered (a thousand trials for each case), the results are summarized in Tables 1 and 2.

In Table 1, the average number of iterations required for generating a stabilizing feedback gain for the six cases is summarized. The results of the PRA and MMA algorithms are
Let $c$! with $a$

A static output feedback controller is to be designed using the Example 3. The following is the state space model of the longitudinal motion of a VTOL helicopter from Keel et al. (1988).

\[
\dot{x} = \begin{bmatrix}
-0.0366 & 0.0271 & 0.0188 & -0.4555 \\
0.0482 & -1.0100 & 0.0024 & -4.0208 \\
0.1002 & 0.3681 & -0.7070 & 1.4200 \\
0 & 0 & 1.0000 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
x \\
x
\end{bmatrix}
\begin{bmatrix}
0.4422 \\
3.5446 \\
-5.5200 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 1.0000 & 0
\end{bmatrix}u,
\]

\[
y = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}x.
\]

A static output feedback controller is to be designed using the ILMI algorithm. After two iterations, $x$ converges to $-0.0512$ with

\[
F = [-0.5586 7.9282]^T.
\]

The eigenvalues of closed-loop system are $-63.2249, -0.1045, -0.2985 \pm 0.9663i$.

Now, we consider the $H_\infty$ control of the above system with

\[
B_1 = \begin{bmatrix}
0 & 0 & 0
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
0 & 0 & 0
\end{bmatrix}, \quad D_{11} = 0,
\]

\[
D_{12} = \begin{bmatrix}
1 & 0
\end{bmatrix}, \quad D_{21} = 1, \quad D_{22} = 0.
\]

Let $\gamma = 1$, after four iterations using our algorithm, we obtain $x = -0.0046, y = -0.0089 \pm 0.4804]$. Let $\gamma = 0.65$, after 43 iterations, we have $x = -9.2828E-5, y = -0.0452 \pm 0.0208]$. Let $\gamma = 0.62$, after 269 iterations, we get $x = -3.3276E-6, y = 0.0074 \pm 0.7832]$. These indicated that a lower bound such that the above system is $H_\infty$ stabilizable via static output feedback is 0.62.

6. Conclusions

In this paper, a new necessary and sufficient condition for the linear time-invariant system to be stabilizable via static output feedback has been given and an iterative LMI algorithm has been presented to compute the stabilizing gain. The algorithm is effective and convergent, though it may fail to determine the feedback gain even it exists. The numerical procedure may be useful to solve this kind of bilinear matrix inequality problem. In this respect, the crucial part is to obtain an iterative condition. When the ILMI algorithm is employed to obtain the stabilizing gain, no extra condition is imposed on the feedback matrix. Therefore, the algorithm can be extended to solve the decentralized stabilization and simultaneous stabilization problems. Note that the algorithm is heuristic and may fail to determine the feedback gain, even it exists.

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