Portfolio selection with mental accounts and background risk

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1. Introduction

Das et al. (2010) develop an appealing model that incorporates features of both behavioral and mean–variance models. Consistent with Shefrin and Statman (2000), Das et al. consider an investor who divides his or her wealth among mental accounts (hereafter, ‘accounts’1 with motives such as retirement and bequest.2 Within each account, the investor seeks to select the portfolio with maximum expected return subject to a constraint that reflects the account’s motive. This constraint precludes the probability that the account’s return is less than or equal to some threshold return from exceeding some threshold probability. Consistent with Markowitz (1952), optimal portfolios within accounts are on the mean–variance frontier. Importantly, Das et al. assume that the investor only faces portfolio risk. In practice, however, many individuals also face background risk. Accordingly, our paper expands upon theirs by considering the case where the investor faces background risk. Our contribution is threefold. First, we provide an analytical characterization of the existence and composition of the optimal portfolios within accounts and the aggregate portfolio. Second, we show that these portfolios lie away from the mean–variance frontier under fairly general conditions. Third, we find that the composition and location of such portfolios can differ notably from those of portfolios on the mean–variance frontier.

1 For an introduction to mental accounting, see, e.g., Thaler (1999) and Nofsinger (2011, Chapters 6 and 7).

2 As Das et al. point out, their model is also consistent with Telser (1955). Das and Statman (2009) expand upon this model by examining optimal portfolios within accounts when derivatives are available.

3 This result assumes that short sales are allowed. In the case where they are disallowed, Das et al. find that the aggregate portfolio lies close to the mean–variance frontier.
Similarly, we next characterize the existence and composition of the aggregate portfolio. The aggregate portfolio exists if and only if the threshold probability of each account is sufficiently low and the threshold return of each account is sufficiently small. Furthermore, we find that the composition of the aggregate portfolio can differ notably from those of portfolios on the mean–variance frontier. Hence, the former portfolio can lie considerably away from this frontier.

The finding that the optimal portfolios within accounts and the aggregate portfolio generally lie away from the mean–variance frontier contrasts with the results of Das et al. who show that such portfolios lie on it. However, we should emphasize that our finding is driven by the presence of background risk, not by mental accounting. Indeed, an investor with a mean–variance objective function who faces background risk and has a single account optimally selects a portfolio that generally lies away from the mean–variance frontier (see, e.g., Baptista, 2008; Jiang et al., 2010). Hence, an investor who faces background risk and has multiple accounts ends up selecting optimal portfolios within accounts and an aggregate portfolio that generally also lie away from it.

There is an extensive literature recognizing the effect of background risk on portfolio selection. For example, some papers provide conditions on utility functions under which the presence of background risk makes investors less willing to bear other risks (see, e.g., Pratt and Zeckhauser, 1987; Kimball, 1993; Gollier and Pratt, 1996). Other papers examine the effect of background risk on the optimal portfolios of investors who use an expected utility model (see, e.g., Heaton and Lucas, 2000). There are also papers that investigate the effect of background risk on the optimal portfolios of investors who use a mean–variance model (see, e.g., Flavin and Yamashita, 2002; Baptista, 2008; Jiang et al., 2010). Our paper differs from this literature in two respects. First, while we consider an investor with multiple accounts, the literature considers an investor with a single account. Second, we assume that for each account the investor seeks to maximize the account’s expected return subject to a constraint that reflects the account’s motive, whereas the literature assumes that the investor maximizes either expected utility or a mean–variance objective function.

We proceed as follows. Section 2 describes the model, and characterizes the optimal portfolios within accounts and the aggregate portfolio when short sales are allowed. Section 3 provides an example to illustrate these portfolios. Section 4 examines the case when short sales are disallowed. Section 5 concludes.

2. The model

Let \( N > 2 \) be the number of risky assets. The \( N \times 1 \) vector of their expected returns is given by \( \mu \) where the \( n \)th entry represents asset \( n \)’s expected return. The \( N \times N \) variance–covariance matrix of asset returns is given by \( \Sigma \) where the entry in row \( n_1 \) and column \( n_2 \) represents the covariance between the returns on assets \( n_1 \) and \( n_2 \). We assume that rank \( (\Sigma) = N \). Like Das et al., we assume that a risk-free asset is not available.

A portfolio is an \( N \times 1 \) vector \( \mathbf{w} \) with \( w_{1} = 1 \), where 1 is the \( N \times 1 \) vector \( [1 \ldots 1] \).\(^5\) The \( n \)th entry of \( \mathbf{w} \) is the weight of asset \( n \) in portfolio \( \mathbf{w} \). Here, a positive (negative) weight represents a long (short) position.\(^6\) Let \( r_{m} \) denote portfolio \( \mathbf{w} \)’s random return. Note that portfolio \( \mathbf{w} \)’s expected return is \( E[r_{m}] = \mathbf{w} \mu \), whereas its variance is \( \sigma^{2}[r_{m}] = \mathbf{w} \Sigma \mathbf{w} \).

In accordance with mental accounting, we consider an investor who views his or her aggregate portfolio as a collection of portfolios within accounts. Hence, like Das et al., we assume that the investor divides his or her wealth among \( M \geq 2 \) accounts. The allocation of the investor’s wealth among these accounts is given by the \( M \times 1 \) vector \( y \) where the \( m \)th entry represents the fraction of wealth in account \( m \).\(^7\)

Unlike Das et al., however, we assume that background risk is present. More specifically, we make two assumptions in regard to background risk. First, we assume that the investor faces it in all accounts.\(^8\) Second, we assume that he or she faces possibly different background risks in different accounts.

It should be stressed that these two assumptions are plausible in practice. For brevity, consider an individual who divides his or her wealth between retirement and bequest accounts. In practice, retirement accounts often have exposures to the stocks of the companies where individuals are employed (see, e.g., Benartzi and Thaler, 2001). Importantly, retirement plan sponsors often restrict employees from eliminating at least part of these exposures (see, e.g., Poterba, 2003; Markowitz et al., 2010). Similarly, in practice, bequest accounts often have exposures to real estate (see, e.g., Raub, 2009). These exposures often involve properties whose values are difficult to insure (e.g., personal residences and investment real estate). Company stock and real estate exposures can thus be viewed as different background risks in different accounts. Hence, it is plausible to assume that the investor in our model: (1) faces background risk in all accounts (our first assumption); and (2) faces possibly different background risks in different accounts (our second assumption).

The \( M \times 1 \) vector of expected values of the background risks is given by \( v \) where the \( m \)th entry represents the expected value of account \( m \)’s background risk. The \( M \times M \) variance–covariance matrix of background risks is given by \( \Omega \) where the entry in row \( m_1 \) and column \( m_2 \) represents the covariance between the background risks of accounts \( m_1 \) and \( m_2 \). Hence, the variance of account \( m \)’s background risk is denoted by \( \Omega_{mm} \).

The \( N \times M \) matrix of covariances between asset returns and background risks is given by \( \Psi \) where the entry in row \( n \) and column \( m \) represents the covariance between asset \( n \)’s return and account \( m \)’s background risk. Therefore, the \( N \times 1 \) vector of covariances between asset returns and account \( m \)’s background risk is denoted by \( \Psi_{m} \).

Fix any given account \( m \in \{1, \ldots, M\} \). If the investor selects portfolio \( \mathbf{w} \) within account \( m \), then the account’s random return is denoted by \( r_{m} \).\(^9\) Furthermore, the account’s expected return and variance are given by \( E[r_{m}] = \mathbf{w} \mu + v_{m} \) and \( \sigma^{2}[r_{m}] = \mathbf{w} \Sigma \mathbf{w} + \Omega_{mm} + 2 \mathbf{w} \Psi_{m} \), respectively.

The optimal portfolio within account \( m \) solves:

\[
\max_{\mathbf{w} \geq 0} \quad \mathbf{w} \mu + v_{m}
\]

s.t.
\[
\mathbf{w} \mathbf{1} = 1
\]

\[
P[r_{m} \leq H_{m}] \leq \alpha_{m},
\]

where \( H_{m} \in \mathbb{R} \) and \( \alpha_{m} \in (0,1/2) \) denote account \( m \)’s threshold return and threshold probability, respectively. Consistent with mental accounting, we follow Das et al. in assuming that the investor selects the portfolio within any given account while ignoring the portfolios within other accounts.

\(^{4}\) An Appendix containing proofs of the theoretical results in our paper can be downloaded from [http://home.gwu.edu/~alexjbapt/JBF4Appendix.pdf](http://home.gwu.edu/~alexjbapt/JBF4Appendix.pdf).

\(^{5}\) Note that we use \( \mathbf{1} \) to transpose a vector.

\(^{6}\) Section 4 examines the case when short sales are disallowed.

\(^{7}\) Following Das et al., we assume that \( y \) is exogenously determined. Note that allowing the investor to endogenously determine \( y \) might be inconsistent with the idea of having multiple accounts. Indeed, this idea breaks down if the investor allocates 100% of his or her wealth to a single account.

\(^{8}\) Nevertheless, our model is more generally applicable when the investor is assumed to face background risk in \( M \) accounts where \( 0 < M \in \mathbb{N} \).

\(^{9}\) Like Das et al., we assume that the investor directly allocates the wealth in each account among available assets. Alexander and Baptista (2011) expand upon Das et al. by assuming that the investor allocates the wealth in each account among portfolio managers, which in turn allocate the wealth under management among available assets.
folio exposures (as well as the background risk exposures) in other accounts (see Eqs. (1)–(3)). In doing so, the investor might end up holding an aggregate portfolio that is sub-optimal from the perspective of investors with mean–variance objective functions (even if background risk is absent as in the model of Das et al.). Nonetheless, we are interested in examining the implications of mental accounting, it assumes that background risk is present. First, there are important reasons for us to examine a model consistent with mental accounting where background risk is present. First, while there is a theoretical literature that explores the portfolio selection implications of mental accounting, it assumes that background risk is absent (see, e.g., Das et al.). As noted earlier, however, individuals often face background risk in practice. Second, there is also an empirical literature suggesting that in practice some individuals indeed behave in accordance with mental accounting (see, e.g., Choi et al., 2009).

We assume that asset returns and background risks have a multivariate normal distribution. Let \( z_a \equiv -\Phi^{-1}(\alpha) \), where \( \Phi(.) \) denotes the standard normal cumulative distribution function. Since \( \alpha \in (0,1/2) \), we have \( z_a > 0 \). Portfolio \( w \)'s Value-at-Risk (VaR) at confidence level 1 – \( \alpha \) is given by:

\[
V[1 - \alpha, r_{w,m}] \equiv z_a \sigma[r_{w,m}] - E[r_{w,m}].
\]

Using Eq. (4), portfolio \( w \) satisfies constraint (3) if and only if:

\[
V[1 - \alpha, r_{w,m}] \leq -H_m.
\]

Therefore, constraint (3) can be thought of as a VaR constraint with confidence level 1 – \( \alpha_m \) and bound – \( H_m \). It follows from Eq. (4) that constraint (5) is equivalent to:

\[
E[r_{w,m}] \geq H_m + z_a \sigma[r_{w,m}].
\]

Hence, the set of portfolios that satisfy constraint (3) has a simple representation in \( (E[r_{w,m}], \sigma[r_{w,m}]) \) space as Fig. 1 illustrates. Portfolios on or above a line with intercept \( H_m \) and slope \( z_a \) satisfy constraint (3), whereas portfolios below it do not. Note that the constraint is tightened if \( H_m \) increases or \( \alpha_m \) decreases.

2.1. Optimal portfolios within accounts

Three portfolios are useful to characterize the optimal portfolio within account \( m \): (1) \( \mathbf{w}_0 \equiv \frac{1}{\mathbf{C}} \mathbf{r}_m \), where \( \mathbf{C} \equiv 1 \mathbf{\Sigma}^{-1} \mathbf{1} \); (2) \( \mathbf{w}_l \equiv \frac{1}{\mathbf{C}} \mathbf{r}_m + \alpha \mathbf{A} \mathbf{r}_0 - \mathbf{1}^T \mathbf{A} \mathbf{1} \); and (3) \( \mathbf{w}_m \equiv (1 + \alpha_m)\mathbf{w}_0 - \mathbf{1}^T \mathbf{A} \mathbf{1} \mathbf{R}_m \), where \( \mathbf{A} \equiv 1 \mathbf{\Sigma}^{-1} \mathbf{1} \mathbf{\Sigma}^{-1} \). Each of these portfolios has a special property. First, portfolio \( \mathbf{w}_0 \) globally minimizes variance (in the absence of background risk). Second, portfolio \( \mathbf{w}_l \) is located in mean–variance space where a ray from the origin intersects the mean–variance frontier (in the absence of background risk) after passing through \( \mathbf{w}_0 \). Third, portfolio \( \mathbf{w}_m \) globally minimizes account \( m \)'s variance (in the presence of the account's background risk).

For brevity, let \( E_m (\sigma^2_m) \) denote account \( m \)'s expected return (variance) if portfolio \( \mathbf{w}_m \) is selected within account. Assuming that \( \alpha_m < \Phi(-\sqrt{D/C}) \), or equivalently, that \( \alpha_m > \sqrt{D/C} \), let

\[
H_m \equiv E_m - \sqrt{(z_{R_m}^2 - D/C)\sigma^2_m},
\]

where \( D = BC - A^2 \) and \( B = \mu^T \mathbf{1} \mathbf{\Sigma}^{-1} \mu \). Letting \( \mathbf{w}_m \) denote the portfolio that minimizes account \( m \)'s VaR at confidence level 1 – \( \alpha_m \) it can be shown that \( V[1 - \alpha_m, r_{w,m}] = -H_m \). It follows that \( H_m \) is closely related to portfolio \( \mathbf{w}_m \)'s VaR at confidence level 1 – \( \alpha_m \). Therefore, as we show shortly, constant \( H_m \) is useful in characterizing the existence of the optimal portfolio within account \( m \).

Theorem 1. Fix any given account \( m \in \{1, \ldots, M\} \). (i) If either (a) \( \alpha_m > \Phi(-\sqrt{D/C}) \), or (b) \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m > H_{\alpha_m} \), then the optimal portfolio within account \( m \) does not exist. (ii) If \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \), then the optimal portfolio within account \( m \) is given by:

\[
\mathbf{w}_m \equiv \mathbf{w}_m + \eta_m (\mathbf{w}_l - \mathbf{w}_0)
\]

where

\[
\eta_m = \frac{E_m - E_n}{B A - A^2/C},
\]

and

\[
E_m = E_n + \sqrt{(D/C)(\sigma^2_m - \sigma^2_l)}.
\]

10 Our model extends to the case where the investor selects the portfolio within any given account while taking into consideration the background risks that he or she faces in all accounts. In this case, instead of assuming that the investor faces different background risks in different accounts, he or she would be assumed to face a single aggregate background risk in all accounts (that would combine the aforementioned background risks). However, such a case is somewhat inconsistent with mental accounting. Note that the investor would select the portfolio within any given account while still ignoring the portfolio exposures in other accounts as in Das et al., but the idea of mental accounting breaks down to some extent since he or she would select this portfolio while taking into consideration the background risk exposures in all accounts. Nevertheless, Section 3.6 illustrates our model when the investor is assumed to do so (i.e., is assumed to face the aggregate background risk in all accounts).

11 In their model, the investor holds such a portfolio only if portfolio restrictions are present (e.g., short sales are disallowed).

12 The assumption that asset returns are normally distributed follows Das et al. However, our normality-based results hold more generally when asset returns and background risks have a multivariate elliptical distribution with finite first and second moments. These results also hold, at least as an approximation, when the distribution of asset returns and background risks is unknown but has finite first and second moments.

13 For an introduction to VaR, see, for example, Hull (2009, Chapter 20).

14 For an examination of the effect of a VaR constraint on portfolio selection with the mean–variance model when an investor has a single account and does not face background risk, see Alexander and Baptista (2004).

15 Here, a portfolio is on the mean–variance frontier if there is no portfolio with the same expected return and a smaller variance.

16 Note that \( \sqrt{D/C} \) is the asymptotic slope of the representation of portfolios on the mean–variance frontier in \( (E[r_{w,m}], \sigma[r_{w,m}]) \) space; see Merton (1972).

17 See Lemma 2 in the Appendix (available at the author’s website). For a characterization of the minimum VaR portfolio in the absence of background risk, see Alexander and Baptista (2004).
Using Theorem 1, the optimal portfolio within account \( m \) exists if and only if threshold probability \( \alpha_m \) is sufficiently low and threshold return \( H_m \) is sufficiently small (i.e., \( \alpha_m < \Phi(\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \)). As we illustrate shortly, this portfolio does not exist if either: (a) the threshold probability is sufficiently high (i.e., \( \alpha_m \geq \Phi(\sqrt{D/C}) \)) since account \( m \)'s expected return is unbounded from above; or (b) the threshold probability is sufficiently low and the threshold return is sufficiently large (i.e., \( \alpha_m < \Phi(\sqrt{D/C}) \) and \( H_m > H_{\alpha_m} \)) since there is no feasible portfolio.

Fig. 2 illustrates the existence (or lack thereof) of the optimal portfolio within account \( m \). The curve represents portfolios that minimize account \( m \)'s variance for various levels of account \( m \)'s expected return. The line has intercept \( H_m \) and slope \( z_{\alpha_m} \). Panels A and B consider the case when \( \alpha_m \geq \Phi(\sqrt{D/C}) \) using two different values of \( H_m \). Regardless of the size of \( H_m \), for any sufficiently large level of account \( m \)'s expected return \( E \), there is a portfolio \( w \) on the curve such that \( E[w,m] = E \). Since \( E \) is sufficiently large, this portfolio lies above the line and thus meets constraint (3). It follows that account \( m \)'s expected return is unbounded from above. Hence, the optimal portfolio within account \( m \) does not exist.

Panels C and D consider the case when \( \alpha_m < \Phi(\sqrt{D/C}) \) using two different values of \( H_m \). In Panel C, we have \( H_m > H_{\alpha_m} \). By construction, all portfolios lie on or to the right of the curve. Since all portfolios also lie below the line, there exists no portfolio that meets constraint (3). It follows that there is no feasible portfolio. Hence, the optimal portfolio within account \( m \) does not exist. In Panel D, we have \( H_m < H_{\alpha_m} \) (more precisely, \( H_m < H_{\alpha_m} \)). Note that account \( m \)'s expected return is bounded from above by the level of \( E[w,m] \) at which the line crosses the top half of the curve. Moreover, there are portfolios that lie above the line and thus meet constraint (3). Therefore, the optimal portfolio within account \( m \) exists.

Fix any given account \( m \in \{1, \ldots, M\} \). Using Theorem 1, the existence of the optimal portfolio within account \( m \) depends on: (i) the set of available assets; (ii) the account's background risk; (iii) threshold probability \( \alpha_m \); and (iv) threshold return \( H_m \). In particular, given the set of available assets and the account's background risk, Theorem 1 implies that the values of \( \alpha_m \) and \( H_m \) should be
the three values that Das et al. (2010, pp. 320–321) use in illustrating such a condition is relatively large. Furthermore, this set contains purposes. Nevertheless, qualitatively similar results would have been obtained if

Therefore, any threshold return

implied risk aversion coefficient.23 The next result identifies the value of this coefficient.

Corollary 1. Fix any account m ∈ {1, . . . , M} with x_m < 0, H_m ≤ H_{x_m}. The optimal portfolio within account m coincides with the optimal portfolio of an investor who faces account m’s background risk and has an objective function given by Eq. (12) where γ_m = \frac{\lambda}{\mu_m}.

Consider an account m with x_m < 0, H_m ≤ H_{x_m}. Corollary 1 establishes that a hypothetical investor with a single account who faces account m’s background risk and has an objective function given by Eq. (12) with γ_m = \frac{\lambda}{\mu_m} would optimally select the optimal portfolio within account m.

Of particular interest is a condition under which the optimal portfolio within a given account is on the mean–variance frontier. Corollary 2 provides such a condition.

Corollary 2. Fix any account m ∈ {1, . . . , M} with x_m < 0, H_m ≤ H_{x_m}. The optimal portfolio within account m is on the mean–variance frontier if and only if \Psi_m = \delta_1 v_1 + \delta_2 v_2 for some constants \delta_1 and \delta_2.

Consider an account m with x_m < 0, H_m ≤ H_{x_m}. Corollary 2 says that the optimal portfolio within account m is on the mean–variance frontier if and only if the vector of covariances between asset returns and account m’s background risk, \Psi_m, is a linear combination of unit vector \mathbf{1} and expected asset

\{w_0 and w_1\} are always on it.22 Note that fund w_m has a weight of 100\% in the optimal portfolio within account m. Therefore, the weights of the other two funds, w_0 and w_1, sum to zero. Specifically, w_0 has a weight of −\eta_m, whereas w_1 has a weight of \eta_m. Eqs. (9)–(11) imply that \eta_m can be determined as follows. First, account m’s standard deviation, \sigma_m, is found by using Eq. (11). Second, account m’s expected return, E_m, is found by using the value of \sigma_m in Eq. (10). Lastly, \eta_m is found by using the value of E_m in Eq. (9).

Fig. 3 illustrates the location of the optimal portfolio within account m in (E[r_{w,m}], \sigma^2[r_{w,m}]) space. Point p_m represents this portfolio. The curve represents portfolios that minimize account m’s variance for various levels of account m’s expected return. The line has intercept H_m and slope x_m. Note that the optimal portfolio within account m is located where this line crosses the top half of the curve.

As we show shortly, if the optimal portfolio within account m exists, then it coincides with the optimal portfolio of a hypothetical investor with a single account who faces account m’s background risk and has an objective function U_m : \mathbb{R} × \mathbb{R}^2 → \mathbb{R} defined by:

U_m[E[r_{w,m}], \sigma^2[r_{w,m}]] = E[r_{w,m}] + \frac{\gamma_m}{2} \sigma^2[r_{w,m}]

(12)

for some risk aversion coefficient \gamma_m. Accordingly, we refer to \gamma_m as the implied risk aversion coefficient.23 The next result identifies the value of this coefficient.

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22 Previous work recognizes that the optimal portfolio of an investor who uses the mean–variance model and faces background risk exhibits three-fund separation (see, e.g., Baptista, 2008; Jiang et al., 2010). Our paper differs from this work in two respects. First, while we consider an investor with multiple accounts, this work considers an investor with a single account. Second, we assume that for each account the investor seeks to maximize the account’s expected return subject to a constraint that reflects the account’s motive, whereas such work assumes that the investor seeks to maximize a mean–variance objective function.

23 Motivation for determining the implied risk aversion coefficient of each account can be found in Das et al. (2010, p. 312). In particular, it is useful to estimate the investor’s attitudes toward risk within a given account from its corresponding VaR-like goal (i.e., its threshold return and probability). Like Das et al., we assume that the investor’s attitudes toward risk possibly depend on the account. For example, the investor might be notably risk-averse in a retirement account and considerably less risk-averse in a bequest account. Hence, different accounts have possibly different implied risk aversion coefficients.
return vector $\mathbf{\mu}$. Since $\Psi_m$ is typically not a linear combination of $1$ and $\mathbf{\mu}$, optimal portfolios within accounts generally lie away from the mean–variance frontier. \footnote{Nevertheless, as noted earlier, the optimal portfolio within account $m$ still lies on the curve that represents the portfolios that minimize the account’s variance for various levels of the account’s expected return; see Fig. 3.}

The intuition for Corollary 2 is as follows.\footnote{Jiang et al. (2010) develop the intuition for the effect of background risk on: (a) the composition of the optimal portfolio of an investor with a single account; and (b) the location and shape of the representation of portfolios on the mean–variance frontier in mean–variance space. In regard to (a), they show that the optimal portfolio in the presence of background risk can be decomposed into two components. The first component is a portfolio on the mean–variance frontier. The second component is self-financing (i.e., asset weights sum to zero) and serves to hedge the investor against background risk. In regard to (b), they show that the presence of background risk shifts the representation of portfolios on the mean–variance frontier in mean–variance space to the right but does not change its shape. See also Baptista (2008).}

As we show shortly, if the optimal portfolio within a given account $m$ generally not on the mean–variance frontier, but the other two funds ($\mathbf{w}_0$ and $\mathbf{w}_1$) are always on it. Note that fund $\mathbf{w}_2$ has a weight of 100% in the aggregate portfolio. Therefore, the weights of the other two funds, $\mathbf{w}_0$ and $\mathbf{w}_1$, sum to zero. Specifically, $\mathbf{w}_0$ has a weight of $-\eta_m$, whereas $\mathbf{w}_1$ has a weight of $\eta_m$.

As we show shortly, the aggregate portfolio coincides with the optimal portfolio of a hypothetical investor with a single account who faces the aggregate background risk and has an objective function given by Eq. (13) with $\gamma = \frac{\alpha}{\eta_m} \gamma_m$ would optimally select the optimal portfolio within account $m$. \footnote{More precisely, fix any account $m \in \{1, \ldots, M\}$ with $\alpha_m < \Phi(-\sqrt{D}/C)$. Then, $H_m = H_{a_m}$, and $\Psi_m = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ for some constants $\delta_1$ and $\delta_2$. Also, suppose that portfolio $\mathbf{w}$ is selected within account $m$. Then, the account’s variance is given by $\sigma^2[\mathbf{w}^{m}] = \mathbf{w}^T \Sigma + 2 \delta_2 \mathbf{w}^{T} \mathbf{\mu} + \sigma^2[\mathbf{w}] + 2 \delta_1 \mathbf{1}^T \mathbf{1} \sigma[\mathbf{w}]$, where the first equality follows from the definition of $\sigma^2[\mathbf{w}]$ and the assumption that $\mathbf{w}^{m} = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$, and the second equality follows from the definitions of $\sigma^2[\mathbf{w}]$ and $\mathbf{1}^T \mathbf{1} \sigma[\mathbf{w}]$ and the fact that $\mathbf{1}^T \mathbf{1} = 1$. Hence, the effect of $\mathbf{w}^{m}$ on $\sigma^2[\mathbf{w}]$ solely depends on $\mathbf{1}^T \mathbf{1} \sigma[\mathbf{w}]$.}

The aggregate portfolio coincides with the optimal portfolio of an investor with a single account who faces the aggregate background risk and has an objective function given by Eq. (13) with $\gamma = \frac{\alpha}{\eta_m} \gamma_m$. \footnote{Recall that if portfolio $\mathbf{w}$ is selected within account $m$, then the account’s variance is $\sigma^2[\mathbf{w}^{m}] = \mathbf{w}^T \Sigma + 2 \delta_1 \mathbf{1}^T \mathbf{1} \sigma[\mathbf{w}]$. The fact that $\mathbf{w}^{m}$ minimizes account $m$’s variance.}

### 2.2. Aggregate portfolio

As noted earlier, the combination of the background risks that the investor faces within all accounts is referred to as aggregate background risk. Let $\mathbf{w}_m$ denote the random return generated by portfolio $\mathbf{w}$ and the aggregate background risk. Also, we refer to $E[\mathbf{w}_m] = \mathbf{w}^T \mathbf{\mu} + y^T \mathbf{\Omega} + \sigma^2[\mathbf{w}] = \mathbf{w}^T \Sigma + y^T \mathbf{\Omega} + 2 \mathbf{w}^{T} \Psi \mathbf{\Psi}^T$ as, respectively, the aggregate expected return and aggregate variance generated by portfolio $\mathbf{w}$ and the aggregate background risk.

The portfolio that globally minimizes aggregate variance is given by $\mathbf{w}_0 = (1 + A_D \mathbf{w}_0 - \Sigma^{-1} \Psi \mathbf{\Omega}) \mathbf{w}$, where $A_D \equiv \sum_{m=1}^M \eta_m a_m$ is a constant and $\Psi = \sum_{m=1}^M \eta_m \mathbf{w}_m$ is the $N \times 1$ vector of covariances between asset returns and aggregate background risk. Portfolio $\mathbf{w}_0 \equiv \sum_{m=1}^M \eta_m \mathbf{w}_m$ is referred to as aggregate portfolio. Portfolio $\mathbf{w}_0$ is useful to characterize the aggregate portfolio.

**Theorem 2.** Suppose that $\mathbf{w}_0 \equiv \Phi(-\sqrt{D}/C)$ and $H_m \leq H_{a_m}$ for any account $m \in \{1, \ldots, M\}$. Then, the aggregate portfolio is given by:

$$\mathbf{w}_0 = \mathbf{w}_0 + \eta_m (\mathbf{w}_1 - \mathbf{w}_0)$$

(14)

where $\eta_m = \sum_{m=1}^M \eta_m$. \footnote{Note that Corollaries 2 and 5 are related. Consider the following two conditions: (1) for any given account $m \in \{1, \ldots, M\}$, $\Psi_m = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ for some constants $\delta_1$ and $\delta_2$; and (2) $\mathbf{w}_0 = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ for some constants $\delta_1$ and $\delta_2$. If condition (1) holds, then condition (2) also holds (note that $\Psi_m = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ where $\delta_1 = \sum_{m=1}^M \eta_m a_m$ and $\delta_2 = \sum_{m=1}^M \eta_m \mathbf{w}_m$). Hence, if the optimal portfolios within accounts lie on the mean–variance frontier (see Corollary 2), then the aggregate portfolio also lies on it (see Corollary 5). However, if condition (2) holds, then condition (1) may not hold. Therefore, the aggregate portfolio might lie on the mean–variance frontier (see Corollary 5) even if the optimal portfolios within accounts lay away from it (see Corollary 2).}
Suppose that $\sigma_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{sa}$ for any account $m \in \{1, \ldots, M\}$. Corollary 5 says that the aggregate portfolio is on the mean–variance frontier if and only if the vector of covariances between asset returns and aggregate background risk, $\mathbf{\Psi}_a$, is a linear combination of unit vector 1 and expected asset return vector $\mathbf{\mu}$. Since $\mathbf{\Psi}_a$ is typically not a linear combination of 1 and $\mathbf{\mu}$, the aggregate portfolio generally lies away from the mean–variance frontier.

The intuition for Corollary 5 is as follows. As noted earlier, the aggregate portfolio exhibits three-fund separation where one of these funds minimizes aggregate variance, and the other two funds are on the mean–variance frontier. Suppose that the vector of covariances between asset returns and aggregate background risk is a linear combination of the unit and expected asset return vectors. Then, the effect of selecting any given aggregate portfolio on aggregate variance solely depends on the portfolio’s expected return and variance. Therefore, the fund that minimizes aggregate variance is on the mean–variance frontier. Since the aggregate portfolio exhibits three-fund separation where each of these three funds is on the mean–variance frontier, such a portfolio is also on it.

As we show shortly, if the aggregate portfolio is on the mean–variance frontier, then it coincides with the optimal portfolio of an investor with a single account who does not face background risk and has an objective function given by Eq. (13) for some risk aversion coefficient $\gamma$. The next result identifies the value of this coefficient.

Corollary 6. Suppose that $\sigma_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{sa}$ for any account $m \in \{1, \ldots, M\}$, and $\mathbf{\Psi}_a = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ for some constants $\delta_1$ and $\delta_2$. The aggregate portfolio coincides with the optimal portfolio of an investor with a single account who does not face background risk and has an objective function given by Eq. (13) where $\gamma = \frac{\delta_1}{\mathbf{1}^\top \mathbf{\Psi}_a}$.

Suppose that $\sigma_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{sa}$ for any account $m \in \{1, \ldots, M\}$, and $\mathbf{\Psi}_a = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}$ for some constants $\delta_1$ and $\delta_2$. Corollary 6 establishes that a hypothetical investor with a single account who does not face background risk and has an objective function given by Eq. (13) with $\gamma = \frac{\delta_1}{\mathbf{1}^\top \mathbf{\Psi}_a}$ would optimally select the aggregate portfolio.

2.3. Practical relevance of results

We should stress that our results are of practical relevance to both investors and financial advisers (hereafter, ‘advisers’). In regard to investors, we highlight the need for them to either: (1) recognize background risk while determining their optimal portfolios within accounts; or (2) instruct others (e.g., advisers) to do the same. In regard to advisers, two aspects are worth noting. First, our results are useful to identify the optimal portfolios of their clients when they have accounts and face background risk. Our model is of particular interest since: (1) as Das et al. suggest, it might be easier for individuals to reveal their attitudes toward risk to advisers by using VaR-like goals (i.e., threshold returns and probabilities) for the accounts instead of objective functions; and (2) as we note earlier, in practice individuals often face background risk.

Second, advisers should be aware that the composition of their clients’ optimal portfolios notably depends on background risk. For example, while these portfolios generally lie away from the mean–variance frontier when clients face background risk, they lie on this frontier when clients do not face it. Also, if two clients have the same goals for a given account, then their optimal portfolios within the account might still differ when they face different background risks.

2.4. Implementing the model

Consider an adviser who seeks to implement our model for a given client. The model requires the adviser to take four steps. First, he or she should identify the sources of background risk that the client faces (e.g., by asking the client to fill out a questionnaire). Second, the adviser should estimate the parameters of the distributions of: (1) asset returns; and (2) background risks. Third, the adviser should elicit the client’s attitudes toward risk with VaR-like goals for his or her accounts that take into consideration these parameters. Fourth, using such parameters and goals, the adviser should determine the client’s optimal portfolios within accounts and the corresponding aggregate portfolio.

3. Example

We now provide an example to illustrate our theoretical results.

3.1. Optimization inputs

Table 1 summarizes the optimization inputs that are used in our example. First, we assume that four assets are available: government bonds, corporate bonds, large stocks, and small stocks. In measuring the returns on government and corporate bonds, we use the Bank of America Merrill Lynch US Treasury and corporate bond indices, respectively. Similarly, in measuring the returns on large and small stocks, we use the S&P 500 and S&P 600 total return stock indices, respectively. Sample return means, variances, and covariances were computed using monthly data during 1995–2010. These statistics were annualized and then were used as optimization inputs. The first two columns of Panel A present the asset expected returns and standard deviations, whereas the last four columns show the correlations between asset returns.

---

32 The Bank of America Merrill Lynch US Treasury index tracks the performance of US dollar denominated sovereign debt publicly issued by the US government in the US. On December 31, 2010, the addition of a given security to the index required that it had: (a) at least one year until it matures; (b) a fixed coupon; and (c) a minimum amount outstanding of $1 billion. Similarly, the Bank of America Merrill Lynch US corporate bond index tracks the performance of US dollar denominated investment grade corporate debt publicly issued in the US. On December 31, 2010, the addition of a given security to the index required that it had: (a) at least one year until it matures; (b) a fixed coupon; and (c) a minimum amount outstanding of $250 million.

33 The S&P 500 stock index tracks the performance of small stocks in the US. On March 31, 2011, the addition of a given security to the index required that it had a market capitalization in excess of $4 billion. Similarly, the S&P 600 stock index tracks the performance of small stocks in the US. On March 31, 2011, the addition of a given security to the index required that it had a market capitalization between $300 million and $1.4 billion.

34 The important issue here is that these optimization inputs are realistic, not how they are estimated.
Second, we assume that the investor has three accounts. Panel B shows their threshold returns and probabilities, and the fractions of the investor’s wealth in these accounts. Panel B presents the expected values and standard deviations of the background risks, and the correlations between background risks. Panel D provides the correlations between asset returns and background risks. Expected returns, standard deviations, threshold returns, threshold probabilities, fractions of wealth, and expected values are reported in percentage points.

### Table 1
Optimization inputs. This table summarizes the optimization inputs that are used in our example. Four assets are available: government bonds, corporate bonds, large stocks, and small stocks. In measuring the returns on government and corporate bonds, we use the Bank of America Merrill Lynch US Treasury and corporate bond indices, respectively. Similarly, in measuring the returns on large and small stocks, we use the S&P 500 and S&P 600 total return stock indices, respectively. Sample return means, variances, and covariances were computed using monthly data during 1995–2010. These statistics were annualized and then were used as optimization inputs. The first two columns of Panel A present the asset expected returns and standard deviations, whereas the last four columns show the correlations between asset returns. The investor has three accounts. Panel B shows their threshold returns and probabilities, and the fractions of the investor’s wealth in the accounts. The investor faces background risk in each account. Panel C provides the correlations between asset returns and background risks, and the fractions of the investor’s wealth in the accounts. Panel D provides the correlations between asset returns and background risks. Expected returns, standard deviations, threshold returns, threshold probabilities, fractions of wealth, and expected values are reported in percentage points.

<table>
<thead>
<tr>
<th>Panel A: Assets</th>
<th>Expected return</th>
<th>Standard deviation</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Government bonds</td>
<td>6.44</td>
<td>4.70</td>
<td>1.00</td>
</tr>
<tr>
<td>(2) Corporate bonds</td>
<td>7.17</td>
<td>5.66</td>
<td>1.00</td>
</tr>
<tr>
<td>(3) Large stocks</td>
<td>9.45</td>
<td>15.99</td>
<td>1.00</td>
</tr>
<tr>
<td>(4) Small stocks</td>
<td>12.24</td>
<td>19.67</td>
<td>1.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Account</th>
<th>Expected value</th>
<th>Standard deviation</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1.00</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>1.00</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

| Panel D: Correlations between asset returns and background risks |
|----------------|----------------|----------------|
| Government bonds | 0.10 | -0.20 |
| Corporate bonds | 0.20 | -0.10 |
| Large stocks | 0.30 | -0.30 |
| Small stocks | 0.10 | -0.10 |

3.2. Existence of optimal portfolios within accounts

Using Theorem 1, the existence of the optimal portfolio within any given account $m \in \{1, 2, 3\}$ depends on: (i) the set of available assets; (ii) the account’s background risk; (iii) the account’s threshold probability $x_{m}$; and (iv) the account’s threshold return $H_{m}$. Of particular interest is the question of existence of such a portfolio given the set of available assets and the account’s background risk. We examine this question in Fig. 4. Panel A considers account 1, whereas Panels B and C consider accounts 2 and 3, respectively. For any given account $m \in \{1, 2, 3\}$, the line plots the value of $H_{m}$ (y-axis) as a function of $x_{m}$ (x-axis). Noting that $\sqrt{D/C} = 0.2967$ with the optimization inputs in Panel A of Table 1, we have $\Phi(-\sqrt{D/C}) = \Phi(-0.2967) = 38.33\%$. Using Theorem 1, the optimal portfolio within account $m$ does not exist if either: (a) $x_{m} > 38.33\%$; or (b) $x_{m} < 38.33\%$ and $H_{m} > H_{\ast}$. However, it follows from Theorem 1 that the portfolio exists if $x_{m} < 38.33\%$ and $H_{m} < H_{\ast}$. In each panel, the dot (*) represents the pair $(H_{m}, x_{m})$ that is used in our example (see the first two rows of Panel B of Table 1). Provided that $x_{m} < 38.33\%$, the optimal portfolio within account $m$ exists if and only if the dot lies on or below the line in the panel that corresponds to this account. As the second row of Panel B of Table 1 shows, we have $x_{m} < 38.33\%$ for any account $m \in \{1, 2, 3\}$. Moreover, since the dot lies below the line in each panel, optimal portfolios within accounts exist in our example.

Consider the case of account 1. Theorem 1 implies that the optimal portfolio within account 1 exists if and only if $x_{1} < \Phi(-\sqrt{D/C}) = 38.33\%$ and $H_{1} \leq H_{\ast}$. As the first column of Panel B of Table 1 indicates, $x_{1} = 5\%$ and $H_{1} = -10\%$. Furthermore, it can be shown that $H_{\ast} = -8.75\%$. Therefore, we have

As before, $H_{\ast}$ is defined as in Eq. (7). Also, recall that $H_{\ast}$ is closely related to the VaR of portfolio $\varsigma_{m}$, which minimizes account $m$’s VaR at confidence level $1 - x_{m}$. Specifically, we have $V[1 - x_{m}, H_{\ast} - H_{m}] = -H_{\ast}$; see Section 2.1.
denote, respectively, the threshold probability and return of account \( m \). Existence of optimal portfolios within accounts. This figure examines the

Fig. 4. Existence of optimal portfolios within accounts. This figure examines the existence of optimal portfolios within accounts in our example. Let \( \alpha_m \) and \( H_m \) denote, respectively, the threshold probability and return of account \( m \), where \( m \in \{1, 2, 3\} \). Panel A considers account 1, whereas Panels B and C consider accounts 2 and 3, respectively.

For any given account \( m \in \{1, 2, 3\} \), the line plots the value of \( H_m \) (y-axis) as a function of \( \alpha_m \) (x-axis). Here, \( H_m \) is defined as in Eq. (7). Also, recall that \( H_m \) is closely related to the VaR of the portfolio that minimizes account \( m \)’s VaR at confidence level \( 1 - \alpha_m \) (see Section 2.1). Noting that \( \sqrt{D/C} = 0.2967 \) with the optimization inputs in Panel A of Table 1, we have \( \Phi(-\sqrt{D/C}) = \Phi(-0.2967) = 38.33\% \). In each panel, the dot (‘.’) represents the pair \((H_m, \alpha_m)\) that is used in our example; see the first two rows of Panel B of Table 1. Provided that \( \alpha_m < 38.33\% \), the optimal portfolio within account \( m \) exists if and only if the dot lies on or below the line in the panel that corresponds to this account. As the second row of Panel B of Table 1 shows, we have \( \alpha_m < 38.33\% \) for any account \( m \in \{1, 2, 3\} \). Moreover, since the dot lies below the line in each panel, optimal portfolios within accounts exist in our example. All numbers in the panels are reported in percentage points.

\[ \alpha_1 = 5\% < 38.33\% = \Phi(-\sqrt{D/C}) \text{ and } H_1 = -10\% < -8.75\% = H_3. \]

It follows from Theorem 1 that the optimal portfolio within account 1 exists. Importantly, there is a relatively large set of values of threshold returns and probabilities for which such a portfolio exists. Specifically, this set includes any pair \((H_1, \alpha_1)\) with \( \alpha_1 < 38.33\% \) that lies on or below the line in Panel A of Fig. 4. Similar results are obtained when we consider the cases of accounts 2 and 3.

3.3. Composition of optimal portfolios within accounts

The first three columns of Panel A of Table 2 characterize optimal portfolios within accounts. The first four rows show their composition. The first column indicates that the optimal portfolio within account 1 involves long positions in government bonds, corporate bonds, and small stocks, and a short position in large stocks. While the total weight of bonds in this portfolio is 85.8\% \([=58.5\% + 27.3\%]\), the total weight of stocks is 14.2\% \([=-53.7\% + 67.9\%]\). The relatively small total weight in stocks can be understood by noting that: (a) government and corporate bonds have smaller expected returns and standard deviations than those of large and small stocks; and (b) the threshold return and probability of account 1 result in a relatively tight VaR constraint as we discuss shortly. Notice that the sizeable short position in large stocks is driven by the fact that the correlation between large stock returns and account 1’s background risk is larger than the correlation between the returns on each of the other three assets and account 1’s background risk; see the first column of Panel D of Table 1.

The second column of Panel A of Table 2 indicates that the optimal portfolio within account 2 involves long positions in government bonds, large stocks, and small stocks, and a short position in corporate bonds. While the total weight of bonds in this portfolio is 14.6\% \([=47.6\% + (-33.0\%)]\), the total weight of stocks is 85.4\% \([=31.4\% + 54.0\%]\). The larger total weight in stocks (relative to that in account 1) can be understood by noting that: (a) government and corporate bonds have smaller expected returns and standard deviations than those of large and small stocks; and (b) the threshold return and probability of account 2 result in a relatively moderately tight VaR constraint as we discuss shortly. Notice that the short position in corporate bonds is driven by the fact that the correlation between corporate bond returns and account 2’s background risk is equal to or larger than the correlation between the returns on each of the other three assets and account 2’s background risk; see the second column of Panel D of Table 1.

The third column of Panel A of Table 2 indicates that the optimal portfolio within account 3 involves a short position in government bonds, and long positions in corporate bonds, large stocks, and small stocks. While the total weight of bonds in this portfolio is -43.5\% \([=-86.2\% + 42.7\%]\), the total weight of stocks is 143.5\% \([=48.5\% + 95.0\%]\). The larger total weight in stocks (relative to that in account 2) can be understood by noting that: (a) government and corporate bonds have smaller expected returns and standard deviations than those of large and small stocks; and (b) the threshold return and probability of account 3 result in a relatively loose VaR constraint as we discuss shortly. Notice that the sizeable short position in government bonds is driven by the fact that the correlation between government bond returns and account 3’s background risk is larger than the correlation between the returns on each of the other three assets and account 3’s background risk; see the last column of Panel D of Table 1.

It is worth emphasizing that the composition of optimal portfolios varies notably across accounts. The differences in the composition of optimal portfolios within accounts are driven by differences in: (a) the threshold returns and probabilities that the

\[ \alpha_m < 38.33\% \]

Here, the tightness of the VaR constraint resulting from the threshold return and probability of a given account is assessed relative to the tightness of the VaR constraints resulting from the threshold returns and probabilities of the other two accounts.
the size of the efficiency losses can be economically significant. In-

The next two rows of Panel A of Table 2 indicate that the ex-
pected return and standard deviation of account 1 are smaller than
those of account 2, which in turn is smaller than those of account
3. The ranking of these expected returns and standard deviations
shows that the use of account 1's threshold return and probability
results in a tighter VaR constraint than the use of those of account
2, which in turn results in a tighter VaR constraint than the use of
those of account 3. Hence, as noted earlier, the threshold return
and probability of account 1 (3) result in a relatively tight (loose)
VaR constraint, whereas the threshold return and probability of
account 2 result in a relatively moderately tight VaR constraint.

It can be shown that for any given account \( m \in \{1, 2, 3\} \), there
are no constants \( \delta_1 \) and \( \delta_2 \) such that
\[
\Psi_m = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}.
\]
Using Corollary 2, optimal portfolios within accounts lie away from
the mean–variance frontier. Of particular interest is the distance of
such portfolios from this frontier. Here, a portfolio’s efficiency loss
is the difference between its standard deviation and that of the
portfolio on the mean–variance frontier with the same expected
return. As the next to last row indicates, optimal portfolios within
accounts have positive efficiency losses. Moreover, the efficiency
loss of the optimal portfolio within account 1 is smaller than that
of the optimal portfolio within account 2, which in turn is smaller
than that of the optimal portfolio within account 3. Importantly,
the size of the efficiency losses can be economically significant. In-

Table 2
Optimal portfolios within accounts and aggregate portfolio. Panel A characterizes optimal portfolios within accounts. The first four rows show their composition. The next two rows present the account expected returns and standard deviations. The last two rows provide the efficiency losses and implied risk aversion coefficients of the optimal portfolios within accounts. Short sales are allowed (disallowed) in the first (second) column. Panel B characterizes the aggregate portfolio. The first four rows show its composition. The next two rows present the aggregate expected return and standard deviation. The last two rows provide the efficiency loss and implied risk aversion coefficient of the aggregate portfolio. Short sales are allowed (disallowed) in the first (second) column. Portfolio weights, expected returns, standard deviations, and efficiency losses are reported in percentage points.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Short sales allowed</th>
<th>Short sales disallowed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Account 1</td>
<td>Account 2</td>
</tr>
<tr>
<td>Government bonds</td>
<td>58.5</td>
<td>47.6</td>
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<tr>
<td>Corporate bonds</td>
<td>27.3</td>
<td>−33.0</td>
</tr>
<tr>
<td>Large stocks</td>
<td>−53.7</td>
<td>31.4</td>
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<tr>
<td>Small stocks</td>
<td>67.9</td>
<td>54.0</td>
</tr>
<tr>
<td>Expected return</td>
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</tr>
<tr>
<td>Standard deviation</td>
<td>12.7</td>
<td>22.1</td>
</tr>
<tr>
<td>Efficiency loss</td>
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<td>2.2</td>
</tr>
<tr>
<td>Implied risk aversion coeff.</td>
<td>4.6</td>
<td>2.2</td>
</tr>
<tr>
<td>Asset</td>
<td>Short sales allowed</td>
<td>Short sales disallowed</td>
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<tr>
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<td>------------------------</td>
</tr>
<tr>
<td></td>
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<tr>
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<td>77.3</td>
<td>70.3</td>
</tr>
<tr>
<td>Efficiency loss</td>
<td>1.8</td>
<td></td>
</tr>
</tbody>
</table>

3.4. Aggregate portfolio

The first column of Panel B of Table 2 characterizes the aggregate portfolio. The first four rows show that this portfolio involves a nega-
tive weight in government bonds, and positive weights in corporate
bonds, large stocks, and small stocks. The size of these weights fol-
lows immediately from the asset weights in optimal portfolios with-
in accounts (see the first three columns of Panel A of Table 2) and the
fractions of the investor’s wealth in the accounts (see the last row of
Panel B of Table 1). For example, the weight in government bonds is
−17.1% \([=(58.5\%) \times (0.2) + (47.6\%) \times (0.3) \times (−86.2\%) \times (0.5)\] .

The next two rows indicate that the aggregate expected return
and standard deviation are larger than those of accounts 1 and 2,
but smaller than those of account 3. These results are mainly driven
by the fact that the expected return and standard deviation of
account 1 are smaller than those of account 2, which in turn are smaller
than those of account 3.

It can be shown that there are no constants \( \delta_1 \) and \( \delta_2 \) such that
\[
\Psi_s = \delta_1 \mathbf{1} + \delta_2 \mathbf{\mu}.
\]
Using Corollary 5, the aggregate portfolio lies away from the mean–variance frontier. Hence, the next to last row indi-
Table 3
Optimal portfolios within accounts and aggregate portfolio when they lie on the mean–variance frontier. Panel A provides correlations between asset returns and background risks for which the optimal portfolios within accounts and the aggregate portfolio lie on the mean–variance frontier when short sales are allowed. Panel B characterizes optimal portfolios within accounts. The first four rows provide their composition. The next two rows present the account expected returns and standard deviations. The last two rows provide the efficiency losses and implied risk aversion coefficients of the optimal portfolios within accounts. Panel C characterizes the aggregate portfolio. The first four rows provide its composition. The next two rows present the aggregate expected return and standard deviation. The last two rows provide the efficiency loss and implied risk aversion coefficient of the aggregate portfolio. Short sales are allowed. Portfolio weights, expected returns, standard deviations, and efficiency losses are reported in percentage points.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Panel A: Correlations between asset returns and background risks</th>
<th>Panel B: Optimal portfolios within accounts</th>
<th>Panel C: Aggregate portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Account 1</td>
<td>Account 2</td>
<td>Account 3</td>
</tr>
<tr>
<td>Government bonds</td>
<td>0.0753</td>
<td>−0.1136</td>
<td>0.1377</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>0.0897</td>
<td>−0.0950</td>
<td>0.1139</td>
</tr>
<tr>
<td>Large stocks</td>
<td>0.0618</td>
<td>−0.0343</td>
<td>0.0398</td>
</tr>
<tr>
<td>Small stocks</td>
<td>0.0800</td>
<td>−0.0286</td>
<td>0.0319</td>
</tr>
<tr>
<td>Government bonds</td>
<td>51.1</td>
<td>28.6</td>
<td>−27.3</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>26.6</td>
<td>42.8</td>
<td>82.9</td>
</tr>
<tr>
<td>Large stocks</td>
<td>−12.0</td>
<td>−22.3</td>
<td>−48.0</td>
</tr>
<tr>
<td>Small stocks</td>
<td>34.3</td>
<td>50.9</td>
<td>92.4</td>
</tr>
<tr>
<td>Expected return</td>
<td>10.3</td>
<td>12.0</td>
<td>15.0</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>12.3</td>
<td>21.1</td>
<td>33.7</td>
</tr>
<tr>
<td>Efficiency loss</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Implied risk aversion coefficient</td>
<td>6.7</td>
<td>4.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Account 1</td>
<td>5.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>59.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large stocks</td>
<td>−33.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small stocks</td>
<td>68.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expected return</td>
<td>100.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>13.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Efficiency loss</td>
<td>21.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied risk aversion coefficient</td>
<td>3.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.5. The case where the investor selects portfolios on the mean–variance frontier

Consider now the case where correlations between asset returns and background risks are given by Panel A of Table 3. This case is of special interest because optimal portfolios within accounts are on the mean–variance frontier as we show shortly.42

42 Note that the set of vectors of correlations between asset returns and the background risk of a given account for which the optimal portfolio within this account lies on the mean–variance frontier is not a singleton. For example, using vectors of correlations equal to the vectors in Panel A of Table 3 multiplied by any constant between −1 and 1 would still result in optimal portfolios within accounts that are on the mean–variance frontier.

Importantly, the composition of these portfolios differs notably from that when they lie away from the mean–variance frontier; compare the first four rows of: (i) Panel B of Table 3; and (ii) Panel A of Table 2. For example, consider the case of account 1. The first column of Panel B of Table 3 indicates that the optimal portfolio within account 1 has the following weights (in parentheses): government bonds (51.1%); corporate bonds (26.6%); large stocks (−12.0%); and small stocks (34.3%). In comparison, the first column of Panel A of Table 2 indicates that this portfolio has the following weights (again, in parentheses): government bonds (58.5%); corporate bonds (27.3%); large stocks (−53.7%); and small stocks (67.9%). Similar results are obtained when we consider the cases of accounts 2 and 3.

Note that the account expected returns and standard deviations are smaller than when optimal portfolios within accounts lie away from the mean–variance frontier; compare the next two rows of: (i) Panel B of Table 3; and (ii) Panel A of Table 2. For example, the first column of Panel B of Table 3 shows that the expected return and standard deviation of account 1 are, respectively, 10.3% and 12.3%, whereas the first column of Panel A of Table 2 shows that they are, respectively, 11.0% and 12.7%.

For any given account m ∈ {1, 2, 3}, there are constants δ1 and δ2 such that \( \Psi_1 = \delta_1 1 + \delta_2 m \). Specifically, in the case of account 1, we have \( \Psi_1 = -0.001 \times 1 + 0.0021 \times m \). Similarly, in the case of account 2, we have \( \Psi_2 = -0.001 \times 1 - 0.001 \times m \). Lastly, in the case of account 3, we have \( \Psi_3 = 0.002 \times 1 - 0.001 \times m \). Using Corollary 2, optimal portfolios within accounts are on the mean–variance frontier. Hence, the next to last row of Panel B of Table 3 indicates that they have zero efficiency losses.

The last row presents the risk aversion coefficients implied by optimal portfolios within accounts. As in the case where such portfolios lie away from the mean–variance frontier, account 1’s implied risk aversion coefficient is larger than that of account 2, which in turn is larger than that of account 3. This result is again driven by the degree of tightness of the VaR constraints resulting from the accounts’ threshold returns and probabilities. Recall that the use of account 1’s threshold return and probability results in a tighter VaR constraint than the use of those of account 2, which in turn results in a tighter VaR constraint than the use of those of account 3.

Panel C of Table 3 characterizes the aggregate portfolio. Note that the composition of the aggregate portfolio differs notably from that when the optimal portfolios within accounts lie away from the mean–variance frontier; compare the first four rows of: (i) Panel C of Table 3; and (ii) Panel B of Table 2. Importantly, noting that \( \Psi_a = 0.0005 \times 1 + 0.0034 \times m \), Corollary 5 implies that the aggregate portfolio lies on the mean–variance frontier. Hence, the aggregate portfolio has a zero efficiency loss, whereas it has a positive efficiency loss when the optimal portfolios within accounts lie away from the mean–variance frontier; compare the next to last row of: (i) Panel C of Table 3; and (ii) Panel B of Table 2.

3.6. The case where the investor is assumed to face the aggregate background risk in all accounts

Suppose now that the investor is assumed to face the aggregate background risk in all accounts (instead of facing different background risks in different accounts).43 When the investor faces the aggregate background risk in account 1, we find that \( H_a = -16.7\% \). Hence, contrasting with our earlier example results, the optimal portfolio within account 1 does not exist if the threshold return is

43 By definition, the aggregate background risk combines the account background risks described in Panels C and D of Table 1. Hence, the investor now selects the portfolio within any given account while taking into consideration all of such risks; see footnote 10.
Table 4
Optimal portfolios within accounts and aggregate portfolio when the investor is assumed to face the aggregate background risk in all accounts. This table considers the case where the investor is assumed to face the aggregate background risk in all accounts (instead of facing different background risks in different accounts). When the investor faces the aggregate background risk in account 1, we find that $H_1 = -16.7\%$. Hence, the optimal portfolio within account 1 does not exist if the threshold return is $H_1 = -10\%$ as in the first column of Panel B of Table 1 (see the first row). Note that the optimal portfolio within account 1 exists if $H_1 < H_a$. Accordingly, we assume that account 1’s threshold return is $H_1 = H_a = -16.7\%$ so that this portfolio exists. When the investor faces the aggregate background risk in accounts 2 and 3, we find that $H_2 = -10.6\%$ and $H_3 = -6.4\%$. Hence, the optimal portfolios within accounts 2 and 3 exist if the threshold returns are, respectively, $H_2 = -15\%$ and $H_3 = -20\%$ as in the last two columns of Panel B of Table 1 (see the first row).

Therefore, we continue to assume that these threshold returns are used. Panel A characterizes optimal portfolios within accounts. The first four rows show their composition. The next two rows present the aggregate expected returns and standard deviations. The last two rows provide the efficiency losses and implied risk aversion coefficients of the optimal portfolios within accounts. Panel B characterizes the aggregate portfolio. The first four rows show its composition. The next two rows present the aggregate expected return and standard deviation. The last two rows provide the efficiency loss and implied risk aversion coefficient of the aggregate portfolio. Short sales are allowed. Portfolio weights, expected returns, standard deviations, and efficiency losses are reported in percentage points.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Account 1</th>
<th>Account 2</th>
<th>Account 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Government bonds</td>
<td>102.3</td>
<td>-13.4</td>
<td>-177.2</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>-68.8</td>
<td>14.2</td>
<td>131.8</td>
</tr>
<tr>
<td>Large stocks</td>
<td>77.7</td>
<td>24.7</td>
<td>50.6</td>
</tr>
<tr>
<td>Small stocks</td>
<td>-11.2</td>
<td>74.5</td>
<td>196.0</td>
</tr>
<tr>
<td>Panel A: Optimal portfolios within accounts</td>
<td>100.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>Expected return</td>
<td>10.9</td>
<td>14.9</td>
<td>20.5</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>16.8</td>
<td>23.3</td>
<td>39.1</td>
</tr>
<tr>
<td>Efficiency loss</td>
<td>4.9</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Implied risk aversion coefficient</td>
<td>9.7</td>
<td>1.8</td>
<td>0.8</td>
</tr>
<tr>
<td>Panel B: Aggregate portfolio</td>
<td>-72.2</td>
<td>56.4</td>
<td>2.4</td>
</tr>
<tr>
<td>Government bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corporate bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large stocks</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small stocks</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expected return</td>
<td>16.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>28.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Efficiency loss</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied risk aversion coefficient</td>
<td>1.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$H_1 = -10\%$ as in the first column of Panel B of Table 1 (see the first row). The non-existence of this portfolio when $H_1 = -10\%$ can be understood by noting that: (1) the investor now faces the aggregate background risk in account 1; and (2) the standard deviation of the aggregate background risk, 17.7%, is relatively large.44 Observe that the optimal portfolio within account 1 exists if $H_1 < H_a$. Accordingly, we assume that account 1’s threshold return is $H_1 = H_a = -16.7\%$ so that this portfolio exists.

When the investor faces the aggregate background risk in accounts 2 and 3, we find that $H_2 = -10.6\%$ and $H_3 = -6.4\%$. Hence, consistent with our earlier example results, the optimal portfolios within accounts 2 and 3 exist if the threshold returns are, respectively, $H_2 = -15\%$ and $H_3 = -20\%$ as in the last two columns of Panel B of Table 1 (see the first row). Therefore, we continue to assume that these threshold returns are used.

Panel A of Table 4 characterizes optimal portfolios within accounts. The first four rows indicate that the composition of such portfolios differs notably from that in the first three columns of Panel A of Table 2. For example, consider the case of account 1. The first column of Panel A of Table 4 shows that the optimal portfolio within account 1 has the following weights (in parentheses): government bonds (102.3%); corporate bonds (−68.8%); large stocks (77.7%); and small stocks (−11.2%). In comparison, as noted earlier, the first column of Panel A of Table 2 shows that this portfolio has the following weights (again, in parentheses): government bonds (58.5%); corporate bonds (27.3%); large stocks (−53.7%); and small stocks (67.9%). The differences in the composition of the optimal portfolio within account 1 are mainly driven by two facts. First, the standard deviation of the aggregate background risk that is used in determining the first column of Panel A of Table 4 differs notably from the standard deviation of account 1’s background risk that is used in determining the first column of Panel A of Table 2.45 Second, the correlations between asset returns and aggregate background risk that are used in determining the former column also differ notably from the correlations between asset returns and account 1’s background risk that are used in determining the latter. Similar results can be seen in the cases of accounts 2 and 3.

Like the first three columns of Panel A of Table 2, the next two columns of Panel A of Table 4 show that the expected return and standard deviation of account 1 are smaller than those of account 2, which in turn are smaller than those of account 3. Unlike the first three columns of Panel A of Table 2, the next to last row of Panel A of Table 4 shows that the efficiency loss of the optimal portfolio within account 1 is larger than that of the optimal portfolio within account 2, which in turn is larger than that of the optimal portfolio within account 3. As in the first three columns of Panel A of Table 2, the last row of Panel A of Table 4 shows that account 1’s implied risk aversion coefficient is larger than that of account 2, which in turn is larger than that of account 3.

Panel B of Table 4 characterizes the aggregate portfolio. The first four rows indicate that the composition of this portfolio differs noticeably from that in the first column of Panel B of Table 2. While the next two rows show that the aggregate expected return and standard deviation are larger than those in the first column of Panel B of Table 2, the last two rows show that the aggregate portfolio’s efficiency loss and its implied risk aversion coefficient are smaller.

4. Disallowing short sales

Like Das et al., we examine the effect of disallowing short sales using our example since no analytical results are available. In doing so, we assume that the correlations between asset returns and background risks are given by Panel D of Table 1.

4.1. Optimal portfolios within accounts

The last three columns of Panel A of Table 2 characterize optimal portfolios within accounts. The results differ from those when short sales are allowed in four main respects (compare the first three columns with the last three columns of the panel). First, while the portfolios when short sales are disallowed involve positions in three or less assets, the portfolios when short sales are allowed involve positions in four assets (see the first four rows). Second, the account expected returns and standard deviations are smaller when short sales are disallowed (see the next two rows). Third, the efficiency losses are smaller when short sales are disallowed, indicating that the optimal portfolios within accounts are closer to the mean–variance frontier (see the next to last row).46 Fourth, the implied risk aversion coefficients of the opti-

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44 In comparison, earlier in our example we assume that the standard deviation of account 1’s background risk is 10% as the second column of Panel C of Table 1 shows (see the first row).

45 Additionally, recall that the value that is used for account 1’s threshold return in Table 4, −16.7%, is different from the one that is used in Table 2, −10%.

46 As before, we compute efficiency losses relative to the mean–variance frontier when short sales are allowed.
mal portfolios within accounts 1 and 2 are larger when short sales are disallowed (see the last row).

4.2. Aggregate portfolio

The second column of Panel B of Table 2 characterizes the aggregate portfolio. The results differ from those when short sales are allowed in four main respects (compare the two columns of the panel). First, while the aggregate portfolio solely involves long positions in three assets when short sales are disallowed, it involves a short position in asset 1 and long positions in the other three assets when short sales are allowed (see the first four rows). Second, the aggregate expected return and standard deviation are smaller when short sales are disallowed (see the next two rows). Third, the aggregate portfolio’s efficiency loss is smaller when short sales are disallowed (see the next to last row). Fourth, one cannot determine the aggregate portfolio’s implied risk aversion coefficient when short sales are disallowed (thus, the last row reports \( E[w_0]\)). The reason why this occurs is that aggregate portfolio \( w_0 \) ends up not minimizing aggregate variance subject to the constraints that: (i) the aggregate expected return is \( E[w_0] \); and (ii) short sales are disallowed.

5. Conclusion

Das et al. (2010) develop an appealing model that incorporates features of both behavioral and mean–variance models. Consistent with Shefrin and Statman (2000), Das et al. consider an investor who divides his or her wealth among mental accounts with different motives such as retirement and bequest. Consistent with Markowitz (1952), optimal portfolios within accounts are on the mean–variance frontier. Therefore, the corresponding aggregate portfolio is also on it.

Importantly, Das et al. assume that the investor only faces portfolio risk. In practice, however, many individuals also face background risk from sources such as labor income and real estate. Indeed, Das et al. suggest an extension of their results to the case where background risk is present. In this paper, we provide such an extension.

Our contribution is threefold. First, we provide an analytical characterization of the existence and composition of the optimal portfolios within accounts and the aggregate portfolio. Second, we show that these portfolios lie away from the mean–variance frontier under fairly general conditions. Third, we find that the composition and location of such portfolios can differ notably from those of portfolios on the mean–variance frontier.

Our results are of practical relevance to both investors and financial advisers. In regard to investors, we highlight the need for them to either: (1) recognize background risk while determining their optimal portfolios within accounts; or (2) instruct others (e.g., financial advisers) to recognize it on their behalf. In regard to financial advisers, two aspects are worth noting. First, our results are useful to identify the optimal portfolios of their clients when they have mental accounts and face background risk. Second, financial advisers should be aware that the composition of their clients’ optimal portfolios notably depends on background risk.

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References


