Portfolio selection problem with liquidity constraints under non-extensive statistical mechanics

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A B S T R A C T

In this study, we consider the optimal portfolio selection problem with liquidity limits. A portfolio selection model is proposed in which the risky asset price is driven by the process based on non-extensive statistical mechanics instead of the classic Wiener process. Using dynamic programming and Lagrange multiplier methods, we obtain the optimal policy and value function. Moreover, the numerical results indicate that this model is considerably different from the model based on the classic Wiener process, the optimal strategy is affected by the non-extensive parameter $q$, the increase in the investment in the risky asset is faster at a larger parameter $q$ and the increase in wealth is similar.

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1. Introduction

The portfolio selection problem is an important and attractive issue in finance. Markowitz [1] was the first to consider the optimal portfolio selection problem and presented the mean-variance approach. This method is a single-period model that makes a one-off decision at the beginning of the period and holds on until the end of the period. Afterwards, Merton [2] extended this single-period model to a continuous-time model by using utility functions and the stochastic control theory.

In real financial markets, to improve risk management, agents often impose some restrictions on their trading, of which the liquidity limit has received much attention from researchers. For example, Xu and Shreve [3] studied a continuous-time portfolio selection problem with a short-selling constraint on a finite horizon and obtained the solution by solving its dual problem. Fu and Lavassani [4] obtained the explicit solutions of the dynamic mean-variance optimal portfolio selection problem with borrowing limits using the stochastic piecewise linear-quadratic control theory. Vila and Zariphopoulou [5] studied an optimal consumption and portfolio selection problem with the borrowing restriction using the stochastic dynamic programming. Luo and Wang [6] studied the portfolio selection problem that occurs when tracking the expected wealth process with liquidity limits and obtained the corresponding Hamilton-Jacobi-Bellman equation with liquidity constraints. Tepla [7] considered an optimal intertemporal portfolio problem with a borrowing limit and short-sale restrictions and introduced an algorithm for its calculation.

However, the above portfolio selection problems were modeled in the framework in which the prices of risky assets were driven by the classical Brownian motions. We know

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that this hypothesis implies that the return distributions of risky assets are normal. However, several empirical results have shown that the returns of financial quantities have the characteristics of fat-tails and aiguilles and are not normal [8–12].

In 1988, Tsallis [13] proposed the non-extensive thermostatistics, which is a generalization of the classical Boltzmann-Gibbs statistics. This non-extensive statistical mechanics was rapidly applied to various fields [14–17]. It has also increasingly drawn attention from finance researchers. For example, Queirós [18] studied characteristics of the return distributions for the Dow Jones and NYSE and found that the $q$-Gaussian distributions derived by the non-extensive statistics mechanics can fit the return distributions in different time scales. Borland [19,20] proposed several option pricing models under the framework of non-extensive statistics mechanics, obtaining some closed-form solutions. A number of studies have shown that the non-extensive statistics methodology has been applied to detect crises of the financial markets. Stavroyiannis and Makris [22] applied the non-extensive statistics methodology to calculate the Value-at-Risk of financial time series and obtained closed-form solutions. A number of studies have shown that the non-extensive statistics methodology has been applied well to the financial field.

In this paper, we establish an optimal portfolio selection model under the framework of the non-extensive statistical mechanics and impose liquidity constraints on it. This article is organized as follows. In Section 2, we model the price process of the risky assets by using Tsallis non-extensive statistical mechanics. The model implicates that the return distributions are $q$-Gaussian distributions with fat-tail characteristics rather than Gaussian ones. In Section 3, we propose an optimal portfolio selection model, which minimizes the cumulative variance between the wealth process and the expected wealth process and subject to a liquidity constraint. In Section 4, we apply the dynamic programming methodology and Lagrange multiplier to solve our optimal portfolio selection problem. In Section 5, we present and discuss the numerical results. In the final section, we summarize the paper.

2. The risky asset price process

Recently, the empirical results have shown that the distributions of stock returns have significant fat-tail characteristics and are not normal distributions. To better fit the fat-tail characteristics of the stock return distribution, we use a stock return fluctuations model, which can be derived from the stochastic processes under the non-extensive statistical framework, to replace the standard Black-Scholes model (see [19,20]). The model is given by

$$dS(t) = S(t)(\mu dt + \sigma d\Omega)$$

where

$$d\Omega(t) = P(\Omega, t)^{1/2}dW(t)$$

$W(t)$ is a Wiener process, $P(\Omega)$ is the Tsallis distribution

$$P(\Omega, t) = \frac{1}{Z(t)}(1 - \beta(t)(1 - q)\Omega^2) \frac{1}{1 - q}$$

with

$$z(t) = ((2 - q)(3 - q)c)^{\frac{1}{q}}$$

$$\beta(t) = c^{\frac{1}{3}}((2 - q)(3 - q)t)^{\frac{1}{3}}$$

and

$$c = \frac{\pi}{q - 1}\frac{\Gamma^2(\frac{1}{q - 1} - \frac{1}{2})}{\Gamma^2(\frac{1}{q - 1})}$$

In the limit $q \to 1$, the Black-Scholes model is recovered. When $1 < q < 5/3$, the distribution exhibits power law tails and has finite variance, which covers the values of empirical returns (see [19]). Hence, this model generalizes the standard Black-Scholes model and can more accurately fit the movements of asset price.

3. Market model and liquidity constraint

Suppose there is a financial market that consists of $n + 1$ assets. One is a risk-free bond whose price process $S_0(t)$ satisfies the following ordinary differential equation

$$\begin{aligned}
    dS_0(t) &= rS_0(t)dt, \quad t \in [0, T] \\
    S_0(0) &= s_0 > 0
\end{aligned}$$

In this equation, the constant $r$ is a positive risk-free rate. The other $n$ assets are stocks whose price processes satisfy the model described in the second section, which is written as the following stochastic differential equation:

$$\begin{aligned}
    dS_i(t) &= S_i(t)(\mu_i dt + \sum_{j=1}^{n} \pi_{ij}d\Omega_j(t)), \quad i = 1, 2, \ldots, n; \ t \in [0, T] \\
    S_i(0) &= s_i > 0
\end{aligned}$$

where

$$d\Omega_j(t) = P(\Omega_j, t)^{1/2}dW_j(t)$$

$W_j(t)$, $j = 1, 2, \ldots, n$ is a Wiener process and $P(\Omega_j, t)$ is the Tsallis distribution of index $q_j$ described in the second section.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ be an $R^n$ valued appreciation rate of returns, $\sigma = [\sigma_{ij}, i, j = 1, 2, \ldots, n]$ be a $n \times n$-matrix valued volatility rate of returns, and $\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_n(t))^T \in L^2_\mathbb{R}([0, T]; R^n)$ be a control process. The component $\pi_i(t)$ is the proportion of the investor’s wealth invested in the $i$th risky asset ($i = 1, 2, \ldots, n$) at time $t$. That is, at time $t$, the agent’s wealth can be given by

$$X^\pi(t) = \sum_{i=0}^{n} \pi_i(t)S_i(t), \ t \geq 0$$

where

$$\pi_0(t) = 1 - \sum_{i=1}^{n} \pi_i(t)$$

Then, the wealth process $\{X^\pi(t)\}$ satisfies the following stochastic differential equation:

$$\begin{aligned}
    dX^\pi(t) &= [rX(t) + (\mu - r)^T\pi(t)]dt + \pi(t)^T\sigma d\Omega(t) \\
    X(0) &= x_0 > 0
\end{aligned}$$
After the insertion of Eq. (9), we can obtain
\[
\begin{align*}
\{dX^\pi(t) &= [rX(t) + (\mu - r)^T \pi(t)] dt + \pi(t)^T \sigma P \pi(t) dW(t) \\
X(0) &= x_0 > 0
\end{align*}
\]
(12)
where \( P_0 = \text{diag}(P_{01}^{1/2}, P_{02}^{1/2}, \ldots, P_{0n}^{1/2})^T \) and \( W(t) = (W_1(t), W_2(t), \ldots, W_n(t))^T \) is a standard \( \mathcal{F}_t \)-adapted \( n \)-dimensional Wiener process.

Without loss of generality, we assume that there is only one risky asset in our financial market, i.e., the case of \( n = 1 \) in the above model. Then, the stochastic differential equation of the wealth process \( X^\pi(t) \) becomes
\[
\begin{align*}
\{dX^\pi(t) &= [rX(t) + (\mu - r) \pi(t)] dt + \pi(t) \sigma P \pi(t) dW(t) \\
X(0) &= x_0 > 0
\end{align*}
\]
(13)

Now, we consider an optimal portfolio selection problem with the liquidity limit. The liquidity limit is such that the funds invested in the risky asset are restricted to a certain proportion of the investor’s total wealth. Moreover, suppose the investor expects the cumulative variance between the wealth process and the expected wealth process to be minimal. Here, the expected wealth process is a function given by the agent according to his subjective objective. Then, the above optimal portfolio selection problem can be formulated as
\[
\begin{align*}
\min_{\pi} & \mathbb{E} \left[ \int_t^T (X^\pi(s) - g(s))^2 ds \right] \\
\text{subject to} & \quad \pi(t) \leq \beta X(t) \\
& \quad (X(\cdot), \pi(\cdot)) \text{ satisfy Eq.(13)}
\end{align*}
\]
where \( g(t) \) is the above expected wealth process. \( 0 < \beta < 1 \) is a constant, which indicates a liquidity limit such that the wealth proportion of the invested risky asset cannot exceed \( \beta \). For example, in China, it is required of insurance companies that the highest ratio of invested stocks is 10 percent. To control the risk of financial markets, an upper bound is also given in other countries.

4. Solution to the optimal portfolio selection problem

Suppose the control \( \pi(t) \) and the underlying dynamics \( X(t) \) remain unchanged throughout the time interval. This assumption is reasonable because in fact, an agent may not continuously change his strategy in a very short time interval. Hence, we can apply the dynamic programming methodology to address our optimal portfolio problem.

Let the value function of our optimization problem be
\[
J(t, x) = \min_{\pi(x(t))} \mathbb{E} \int_t^T (X(v) - g(v))^2 dv | X(t) = x
\]
(15)

By using Itô lemma, we obtain
\[
\begin{align*}
dJ(t, x) &= (J_t + J_x (rx + (\mu - r) \pi) + \frac{1}{2} J_{xx} \pi^2 \sigma^2 2^{1-q} dt \\
&\quad + J_x \pi \sigma P \pi^2 dW(t)
\end{align*}
\]
(16)

Then, the corresponding Hamilton-Jacobi-Bellman equation can be written as
\[
\begin{align*}
J_t + r J_x + \min_{\pi} \left\{ (\mu - r) \pi J_x + \frac{1}{2} \pi^2 \sigma^2 2^{1-q} J_{xx} \right\} + (x - g(t))^2 &= 0 \\
J(T, x) &= 0
\end{align*}
\]
(17)

Under the condition that the time is \( t \) and wealth \( X(t) = x \), our original optimization problem becomes a static optimization problem
\[
\begin{align*}
\min_{\pi} \left\{ (\mu - r) \pi J_x + \frac{1}{2} \pi^2 \sigma^2 2^{1-q} J_{xx} \right\} \\
\text{s.t.} & \quad \pi(t, x) \leq \beta x
\end{align*}
\]
(18)

Now, we can apply the Lagrange multiplier method to address the above static optimization problem with a constraint. Then, the Lagrange function is structured as follows:
\[
L(\pi, \lambda, t, x) = J_x (\mu - r) + \frac{1}{2} J_{xx} \pi^2 \sigma^2 2^{1-q} + \lambda (\beta x - \pi)
\]
(19)

In the equation, \( \lambda \) is a Lagrange multiplier. We can give the first order necessary condition of the above Eq. (19) without difficulty as follows:
\[
\begin{align*}
J_x (\mu - r) + J_{xx} \pi \sigma^2 2^{1-q} - \lambda &= 0 \\
\lambda (\beta x - \pi) &= 0 \\
\lambda &\leq 0
\end{align*}
\]
(20)

Solving the first equation in the Eq. (20), we get
\[
\pi^* = \frac{\lambda - (\mu - r) J_x}{J_{xx} \pi \sigma^2 2^{1-q}}
\]
(21)

Solving the second equation in the Eq. (20), we have
\[
\lambda = 0 \quad \text{or} \quad \lambda < 0 \quad \text{and} \quad \beta x - \pi = 0
\]

Substituting into Eq. (21), respectively, we get the optimal strategy as follows:
\[
\pi^* = \begin{cases} 
\frac{(\mu - r) J_x}{\sigma^2 2^{1-q} J_{xx}} & \lambda = 0 \\
\frac{\lambda}{\beta x} & \lambda < 0
\end{cases}
\]
(22)

Substituting the optimal strategy Eq. (22) into Eq. (17), we obtain the value function \( J \) that satisfies
\[
\begin{align*}
J_t + r J_x + (\mu - r) \pi^* J_x + \frac{1}{2} \pi^* \sigma^2 2^{1-q} J_{xx} + (x - g(t))^2 &= 0 \\
J(T, x) &= 0
\end{align*}
\]
(23)

Eq. (23) is a complicated nonlinearity equation. It is very difficult to work out the explicit solution. However, if there is no liquidity constraint on Eq. (14), we can obtain the explicit solution. In that case, the optimal strategy Eq. (22) becomes
\[
\pi^* = \frac{(\mu - r) J_x}{\sigma^2 2^{1-q} J_{xx}}
\]
(24)

Substituting Eq. (24) into Eq. (23), we have
\[
\begin{align*}
J_t + r J_x - \frac{1}{2} (\mu - r) \sigma^2 2^{1-q} J_{xx} + (x - g(t))^2 &= 0 \\
J(T, x) &= 0
\end{align*}
\]
(25)

Theorem 1. The solution to the optimization problem given by the Hamilton-Jacobi-Bellman Eq. (25) is
\[
J(t, x) = f_0(t) + f_1(t) x + f_2(t) x^2
\]
(26)
where

\[
\begin{align*}
    f_0 &= -\int_t^T \left( \frac{\mu - r}{2\sigma^2 P^{1/2}} \right)^2 \frac{f_1^2(v)}{r^2} (v) + g^2(v) \right) \, dv \\
    f_1 &= -2e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \int_t^T \left( e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} g(v) \right) \, dv \\
    f_2 &= \frac{1}{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \left( e^{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} (T-t) - 1 \right)
\end{align*}
\]

**Proof of Theorem 1**

Try the function

\[
J(t, x) = f_0(t) + f_1(t)x + f_2(t)x^2
\]

and then solve for \( f_0(t), f_1(t), f_2(t). \)

Substituting Eq. (27) into Eq. (25), after a simple derivation, we can get three ordinary differential equations, as follows:

\[
\begin{align*}
    \frac{\partial f_0}{\partial t} - \left( \frac{\mu - r}{2\sigma^2 P^{1/2}} \right)^2 \frac{f_1^2}{r^2} + g^2 &= 0 \\
    f_0(T) &= 0 \\
    \frac{\partial f_1}{\partial t} + \left( r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2 \right) f_1 - 2g &= 0 \\
    f_1(T) &= 0 \\
    \frac{\partial f_2}{\partial t} + \left( 2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2 \right) f_2 + 1 &= 0 \\
    f_2(T) &= 0.
\end{align*}
\]

Without difficulty, from Eq. (28), we have

\[
\begin{align*}
    f_0 &= -\int_t^T \left( \frac{\mu - r}{2\sigma^2 P^{1/2}} \right)^2 \frac{f_1^2(v)}{r^2} (v) + g^2(v) \right) \, dv \\
    f_1 &= -2e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \int_t^T \left( e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} g(v) \right) \, dv \\
    f_2 &= \frac{1}{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \left( e^{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} (T-t) - 1 \right)
\end{align*}
\]

From Eq. (29), we get

\[
f_1 = -2e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \int_t^T \left( e^{-\left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} g(v) \right) \, dv
\]

From Eq. (30), we have

\[
f_2 = \frac{1}{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \left( e^{2r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} (T-t) - 1 \right)
\]

**5. Numerical results**

To test our model, we suppose that an agent’s expected wealth process is \( g(t) = Ae^{\gamma t}, \) where \( \gamma \) is the agent’s expected return. Then, from Eq. (32), we have

\[
f_1 = -\frac{2Ae^{\gamma t}}{\gamma + r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} \left( e^{\gamma + r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} (T-t) - 1 \right)
\]

From Eq. (24), we get

\[
\pi^* = \frac{r - \mu}{\sigma^2 P^{1/2}} \left( x + \frac{f_1}{2f_2} \right)
\]

Substituting the functions \( f_1 \) and \( f_2 \) into Eq. (35), we can obtain the explicit solution

\[
\pi^* = \frac{r - \mu}{\sigma^2 P^{1/2}} \left( x - Ae^{\gamma t} \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2 \left( e^{\gamma + r - \left( \frac{\mu - r}{\sigma^2 P^{1/2}} \right)^2} (T-t) - 1 \right) \right)
\]

Let \( T = 1, t = 0.5, \mu = 0.12, \sigma = 0.2, r = 0.1, \gamma = 0.12, \beta = 0.3, A = 1. \) In the following Fig. 1, we present a comparison of the optimal portfolio versus the wealth without the liquidity constraint between \( q = 1 \) (Black-Scholes model) and \( 1 < q < 5/3. \) In Fig. 2, we give it with the liquidity constraint.

Fig. 1 shows that when the liquidity is not restricted, the proportion invested in the risky asset linearly decreases as the wealth increases at a given time under the Black-Scholes model. However, the decrease in the proportion invested in the risky asset becomes active; afterwards, the increase in the investment in the risky asset is faster at larger values of the non-extensive parameter \( q. \)

Then, we select daily closing prices of the Shanghai Composite Index as experimental datasets. The period is from 01/04/2011 to 01/05/2015 and the yields are the logarithmic returns.

Table 1 shows that the kurtosis coefficient of the daily returns of the Shanghai Composite Index is 5.0329, which is...
0.00017  0.0113  5.0329  167.9051  0.0000

greater than that of the normal density distribution (the kurtosis coefficient of the normal density distribution is 3). From the 4th and 5th columns, we note that the Jarque–Bera test rejects the null hypothesis that the distribution of these daily returns of the Shanghai Composite Index is normal.

By calculating, we find that the q-Gaussian distribution of the parameter $q = 1.38$ can more accurately fit the empirical density distribution of the daily returns than the normal distribution which is presented in Fig. 3.

Fig. 4 shows that the increase per unit time of the wealth under our model is greater than that under the Black-Scholes model. Moreover, under the Black-Scholes model, the value of final wealth is only 1.073, while it is 1.092 under our model.

6. Summary

In summary, we proposed an optimal portfolio selection model with a liquidity limit. In our model, the risky asset price was driven by a new process based on the recent non-extensive statistical mechanics, which can describe fat-tailed characteristics of returns, instead of the classic Wiener process. Moreover, the general model based on Brownian motion was recovered by ours as a special case ($q = 1$).

As future work, we will study optimal portfolio selection problems with other constraints, such as the Value-at-Risk limit, in the framework of the non-extensive statistical mechanics.

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